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$\sup \times \inf$ Inequalities for the Scalar Curvature Equation in Dimensions 4 and 5

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Abstract. We consider the following problem on bounded open set Ω of $\mathbb{R}^n$:

$$\begin{cases} -\Delta u = V u^{\frac{n+2}{n-2}} & \text{in } \Omega \subset \mathbb{R}^n, \quad n = 4,5, \\ u > 0 & \text{in } \Omega. \end{cases}$$

We assume that :

$$\begin{aligned} & V \in C^{1,\beta}(\Omega), && 0 < \beta \leq 1, \\ & 0 < a \leq V \leq b < +\infty, && \\ & |\nabla V| \leq A, \quad |\nabla^{1+\beta} V| \leq B && \text{in } \Omega. \end{aligned}$$

Then, we have a $\sup \times \inf$ inequality for the solutions of the previous equation, namely:

$$\begin{aligned} & \Big(\sup_K u\Big)^{\beta} \times \inf_{\Omega} u \leq c = c(a,b,A,B,\beta,K,\Omega) && \text{for } n = 4, \\ & \Big(\sup_K u\Big)^{1/3} \times \inf_{\Omega} u \leq c = c(a,b,A,B,K,\Omega) && \text{for } n = 5 \quad \text{and} \quad \beta = 1. \end{aligned}$$

Key Words: $\sup \times \inf$, dimension 4 and 5, blow-up, moving-plane method.

AMS Subject Classifications: 35J61, 35B44, 35B45, 35B50

1 Introduction and main result

We work on $\Omega \subset\subset \mathbb{R}^4$ and we consider the following equation:

$$\begin{cases} -\Delta u = V u^{\frac{n+2}{n-2}} & \text{in } \Omega \subset \mathbb{R}^n, \quad n = 4,5, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{E}$$

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with

$$\begin{cases} V \in C^{1,\beta}(\Omega), \\ 0 < a \leq V \leq b < +\infty & \text{in } \Omega, \\ |\nabla V| \leq A & \text{in } \Omega, \\ |\nabla^{1+\beta} V| \leq B & \text{in } \Omega. \end{cases} \qquad (C_\beta)$$

Without loss of generality, we suppose $\Omega = B_1(0)$ the unit ball of $\mathbb{R}^n$.

The corresponding equation in two dimensions on open set Ω of $\mathbb{R}^2$ is:

$$-\Delta u = V(x)e^u. \qquad (E')$$

Eq. (E') was studied by many authors and we can find very important result about a priori estimates in [8, 9, 12, 16, 19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c\Big(\inf_\Omega V,\ ||V||_{L^\infty(\Omega)},\ \inf_\Omega u,\ K,\ \Omega\Big).$$

And, precisely, in [8,12,16,19], we have:

$$C \sup_K u + \inf_\Omega u \leq c = c\Big(\inf_\Omega V, ||V||_{L^\infty(\Omega)}, K, \Omega\Big),$$

$$\sup_K u + \inf_\Omega u \leq c = c\Big(\inf_\Omega V, ||V||_{C^\alpha(\Omega)}, K, \Omega\Big),$$

where K is a compact subset of Ω, C is a positive constant which depends on $\frac{\inf_\Omega V}{\sup_\Omega V}$, and, $\alpha \in (0, 1]$.

For $n \geq 3$ we have the following general equation on a Riemannian manifold:

$$-\Delta u + hu = V(x)u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad (E_n)$$

where h, V are two continuous functions. In the case $c_n h = R_g$ the scalar curvature, we call V the prescribed scalar curvature. Here c_n is a universal constant.

Eq. (E_n) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2–4,11,15]. In this case we have a sup $\times$ inf inequality.

In the case $V \equiv 1$ and M compact, Eq. (E_n) is Yamabe equation. T. Aubin and R. Schoen proved the existence of solution in this case, see for example [1,14] for a complete and detailed summary.

When M is a compact Riemannian manifold, there exist some compactness result for Eq. (E_n) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeormorfic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose M Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [17] proved that the product sup $\times$ inf is bounded. Also, see [3,5,6] for other