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Convergence Analysis and Error Estimate for Distributed Optimal Control Problems Governed by Stokes Equations with Velocity-Constraint

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Abstract. In this paper, spectral approximations for distributed optimal control problems governed by the Stokes equation are considered. And the constraint set on velocity is stated with L^2-norm. Optimality conditions of the continuous and discretized systems are deduced with the Karush-Kuhn-Tucker conditions and a Lagrange multiplier depending on the constraint. To solve the equivalent systems with high accuracy, Galerkin spectral approximations are employed to discretize the constrained optimal control systems. Meanwhile, we adopt a parameter λ in the pressure approximation space, which also guarantees the inf-sup condition, and study a priori error estimates for the velocity and pressure. Specially, an efficient algorithm based on the Uzawa algorithm is proposed and its convergence results are investigated with rigorous analyses. Numerical experiments are performed to confirm the theoretical results.

AMS subject classifications: 49J20, 65N15, 65N35

Key words: Optimal control, spectral approximation, Stokes equation, convergence analysis.

1 Introduction

In recent years, there has been extensive research on theoretical and numerical results of optimal control problems governed by partial differential equations, and most of them are solved using finite element methods, see [1, 8, 15, 21, 23–25] and the references cited therein. The authors in [20] employed finite element approximations to simulate the solutions of the optimality system, and derived optimal error estimates. In [19], the authors

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designed an efficient alternating direction method of multipliers for solving the optimal control problem problem by finite element methods. In [35], the authors derived optimal orders of convergence of state and adjoint state variables with mixed finite element methods. A penalized Neumann boundary control approach for optimal Dirichlet boundary control problems associated with steady-state Navier-Stokes equations was illustrated in [22]. The authors in [16] established the existence and first-order optimality condition of the optimal control with Navier-Stokes equations, and gave a convergence result on the augmented Lagrangian method for non-smooth cost functional. The authors in [38] employed finite element approximations to solve optimal control problems governed by time fractional diffusion equations. Furthermore, fully discrete schemes for time fractional optimal control problems were stated in [39].

Nowadays, both spectral methods and finite element methods are widely used for solving these problems. Further more, the literature about this topic is too huge to summarize. It is well-known that the spectral method provides a high accurate simulation if the solution is smooth enough [6]. Spectral methods for control-constrained optimal control problems were studied in [10, 11]. Mixed spectral methods were proposed to solve the state-constrained optimal control problems in [36]. In engineering applications, optimal control systems have been used to describe the hydrodynamic models, and many research has been devoted to discussing optimal control problems in fluid dynamics. Stokes equations depict the motions of incompressible viscous fluid flow with low Reynolds numbers [14, 28] and the references cited therein. While the a posteriori error estimates were studied with the constraint on control [26], the a priori error estimates and a posteriori error estimates were investigated with the constraint on state [29,30]. Meanwhile, the a priori estimates and a posteriori error estimates of spectral approximations were stated in [9,37]. For lots of computational fluid dynamics, one often focuses on how to control the L^2-norm of velocity. In this paper, we adopt stationary Stokes equations to distributed optimal control problems and select the L^2-norm constraint on the velocity. To simplify the analysis and design of the system, we set the L^2-norm of velocity is not more than a given positive constant. Furthermore, we derive the equivalent optimality conditions and a priori error estimates in details. We also employ the Uzawa algorithm to design an efficient iterative algorithm. Meanwhile, we investigate the convergence of the algorithm with rigorous analyses.

The outline of this paper is organized as follows. In Section 2, we introduce the optimal control model and employ the Karush-Kuhn-Tucker conditions to investigate the first-order equivalent optimality conditions for the continuous systems. In Section 3, we give the Galerkin spectral approximations for the corresponding equivalent weak systems. In Section 4, we deduce the a priori error estimates for the spectral approximations with the help of two orthogonal projections and the Ladyzhenskaya-Babuška-Brezzi (LBB, or the inf-sup)-condition. Section 5 is devoted to designing an efficient algorithm to solve the coupled system. Meanwhile, the convergence of the given algorithm is proved. In Section 6, some numerical experiments are listed to validate the theoretical results. Finally, the conclusions are listed to summarize this paper.