Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **13**, No. 6, pp. 1501-1519

DOI: 10.4208/aamm.OA-2019-0270
December 2021

A Parallel Finite Element Algorithm for the Unsteady Oseen Equations

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Received 12 September 2019; Accepted (in revised version) 15 December 2020

Abstract. Based on fully overlapping domain decomposition, a parallel finite element algorithm for the unsteady Oseen equations is proposed and analyzed. In this algorithm, each processor independently computes a finite element approximate solution in its own subdomain by using a locally refined multiscale mesh at each time step, where conforming finite element pairs are used for the spatial discretizations and backward Euler scheme is used for the temporal discretizations, respectively. Each subproblem is defined in the entire domain with vast majority of the degrees of freedom associated with the particular subdomain that it is responsible for and hence can be solved in parallel with other subproblems using an existing sequential solver without extensive recoding. The algorithm is easy to implement and has low communication cost. Error bounds of the parallel finite element approximate solutions are estimated. Numerical experiments are also given to demonstrate the effectiveness of the algorithm.

AMS subject classifications: 68W10, 65M15, 76M10, 76D05

Key words: Oseen equations, finite element, overlapping domain decomposition, backward Euler scheme, parallel algorithm.

1 Introduction

In the last decades, based on the idea of Xu and Zhou [1–3] for local and parallel finite element discretizations, some new local and parallel algorithms have been proposed for eigenvalue problems [3–5], the stationary incompressible magnetohydro dynamicsthe [6], the steady Stokes equations [7–10], the Stokes/Darcy problem [11], the steady Navier-Stokes equations [12–25] and the stream function form of Navier-Stokes equations [26]. These algorithms have less communication complexity than current standard approaches and allow existing sequential PDE codes to run in a parallel environment

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without a large investment in recoding. It is shown by numerical experiments and comparing results that these algorithms are highly efficient and easy to implement.

Our ultimate goal is to study local and parallel finite element discretization algorithms for the unsteady incompressible Navier-Stokes equations. By linearizing the incompressible Navier-Stokes equations (e.g., by a semi-implicit iteration), we get an Oseen problem, which shows up as an auxiliary problem in many numerical approaches for solving the Navier-Stokes equations. Therefore, it is reasonable to study the Oseen problem first. For the Oseen problem, some numercial methods were proposed and analyzed such as the weak Galerkin finite element methods [27], the Nitsche cut finite element methods [28], the stabilized Nitsche cut finite element methods [29] and the new streamline diffusion finite element methods [30]. However, to the authors' best knowledge, there is no study on local and parallel finite element discretizations for the unsteady Oseen problems.

We notice that local and parallel finite element algorithms for the unsteady Stokes and Navier-Stokes equations were investigated numerically in [31] and [32], respectively. However, there is a lack of theoretical analysis in the above works. In this paper, we extend our work for the unsteady Stokes equations in [33] to the unsteady Oseen problem. Compared with the Stokes equations, the Oseen equations possess a convective term $b\cdot\nabla u$, which has a significant impact on the theoretical analysis and numerical computation. With backward Euler scheme for the temporal discretizations, we develop a parallel fully discrete finite element algorithm, in which each processor independently computes a finite element approximation solution in its own subdomain by using locally refined multiscale mesh at each time step. Meanwhile, using the theoretical tool of local a priori error estimate for the finite element discretization solutions, error bounds of the obtained finite element approximations from our parallel algorithm are derived.

The outline of the paper is as follows. In the next section, some mathematical preliminaries and assumptions on the mixed finite element spaces are given. In Section 3, local finite element method is given and error estimates are derived, which play an important role in analysing the proposed algorithms in the next section. In Section 4, based on fully overlapping domain decomposition technique, a parallel finite element method for solving the unsteady Oseen equation is proposed. Numerical experiments are given to verify the effectiveness of the algorithm in Section 5. Finally, some conclusions in Section 6 are obtained.

2 Mathematical preliminaries

Let Ω be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ in $\mathbb{R}^2$ and $(0,T)$ be a time interval with $T<\infty$. we shall use the standard notations $W^{m,p}(\Omega)$ for Sobolev spaces and their associated norms $\|\cdot\|_{m,p}$ and seminorms $|\cdot|_{m,p}$ (cf. [34–36]). For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$ and then denote by $H_0^1(\Omega)$ the closed subspace of $H^1(\Omega)$ consisting of functions with zero trace on $\partial\Omega$. The space $H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$ and its associated