

An ADI Sparse Grid method for Pricing Efficiently American Options under the Heston Model

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Received 13 October 2020; Accepted (in revised version) 9 March 2021

Abstract. One goal of financial research is to determine fair prices on the financial market. As financial models and the data sets on which they are based are becoming ever larger and thus more complex, financial instruments must be further developed to adapt to the new complexity, with short runtimes and efficient use of memory space. Here we show the effects of combining known strategies and incorporating new ideas to further improve numerical techniques in computational finance.

In this paper we combine an ADI (alternating direction implicit) scheme for the temporal discretization with a sparse grid approach and the combination technique. The later approach considerably reduces the number of "spatial" grid points. The presented standard financial problem for the valuation of American options using the Heston model is chosen to illustrate the advantages of our approach, since it can easily be adapted to other more complex models.

AMS subject classifications: 65M10, 78A48

Key words: Sparse grid, combination technique, American options, ADI, Heston model.

1 Introduction

A fair price of a financial derivative is arbitrage-free, which means that the price does not guarantee a profit. A financial derivative is a contract between parties whose value at the maturity date T is determined by the underlying assets at the time T or before the time T . Options are a special type of financial derivative.

A plain vanilla option is a contract that gives the holder the right (but not the obligation) to exercise a particular transaction at time T or until time T at a fixed price K (strike). We distinguish between call and put options. A call option holder has the right to buy

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from the writer, and if a put option is held, the holder has the right to sell it to the writer. The time of exercise defines the type of option: if the holder has the right to exercise the option only on a certain predefined expiration date T , a European option is used, whereas if the holder can exercise at any time before and at maturity T , an American option is considered. In addition to European and American plain vanilla put and call options, there are other types of options that take into account different trading strategies [8].

In our paper we focus on American options. The holder of a put option exercises the option if $S < K$, since he can sell the predefined amount at the price $K \in \mathbb{R}^+$ instead of the market price for the underlying $S \in \mathbb{R}^+$. The exercise region of a put option is defined as the range in which a profit is gained, this is the region where $K - S > 0$. If $K - S \leq 0$ the option will not be exercised, because exercising it would result in a loss. Similarly, a call option will be exercised in the $K < S$ region. These results are summarized in the payoff-function $\phi(S)$:

$$\phi(S) = \begin{cases} \max(S - K, 0) = (S - K)^+ & \text{for } S \geq 0 \quad (\text{Call}), \\ \max(K - S, 0) = (K - S)^+ & \text{for } S \geq 0 \quad (\text{Put}), \end{cases} \quad (1.1)$$

with the abbreviation $(\cdot)^+ = \max(\cdot, 0)$. Since American options can be exercised before the maturity, the trading strategy is to exercise the option on the unknown time point before or at maturity, where $K - S > 0$ is maximal. Therefore the time dependent free boundary value $S_f(t)$ is introduced and for the price of a American Put option $P(S, t)$ holds

$$\begin{aligned} P(S, t) &= \phi(S) = (K - S)^+ = K - S && \text{for } S \leq S_f(t), \\ P(S, t) &> \phi(S) = (K - S)^+ && \text{for } S > S_f(t). \end{aligned}$$

The dynamics of the price of the underlyings can be described via a stochastic differential equation (SDE) which corresponds to a partial differential equation (PDE). In 1973 Black and Scholes developed the Black-Scholes model [1], where the dynamic is described by

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^S, \quad (1.2)$$

where r is the constant interest rate, σ is the constant variance and dW_t^S denotes a Brownian motion. Starting from this SDE, we obtain the price of an American Put Option $P(S, t)$ by solving the partial differential equation

$$P(S, t) = \frac{\partial V}{\partial t} + \frac{1}{2} \nu S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \geq 0, \quad S > 0, \quad 0 < t \leq T. \quad (1.3)$$

Since ν and r are non-constant parameters in the real market as well, until now several extension have been developed to gain more flexibility and comparability to real market situations. Some extensions consider nonlinear functions, e.g., for σ resulting in nonlinear Black-Scholes models [6], other extensions include an additional SDE, e.g., a stochastic volatility or a stochastic interest rate [11, 27]. In the sequel we discuss the Heston