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Asymptotic Analysis and a Uniformly Convergent Numerical Method for Singular Perturbation Problems

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Abstract. Approximation methods for boundary problems for a fourth-order singularly perturbed partial differential equation (PDE) are studied. Using a suitable variable change, we reduce the problem to a second-order PDE system with coupled boundary conditions. Taking into account asymptotic expansions of the solutions, we discrete the resulting problem by a tailored finite point method. It is proved that the scheme converges uniformly with respect to the small parameter involved. Numerical results are consistent with the theoretical findings.

AMS subject classifications: 65N35, 35C20

Key words: Tailored finite point method, singular perturbation problem, asymptotic analysis.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $0 < \epsilon \ll 1$. In this work, we focus on a fourth-order equation with two different boundary conditions — viz.

$$\begin{aligned} &\epsilon^2\Delta^2 u - \Delta u = f \quad &&\text{in } \Omega,\\ &\Delta u = u = 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\epsilon^2\Delta^2 u - \Delta u = f \quad &&\text{in } \Omega,\\ &\frac{\partial u}{\partial \mathbf{n}} = u = 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where f is a piecewise smooth function.

Equations with a small positive factor ϵ in front of the highest order derivative are called singular perturbation equations (SPEs). They attracted substantial attention and are often exploited in multi-scale modeling of various problems in structural mechanics, fluid

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mechanics, and reaction-diffusion processes — cf. [3, 4, 33]. In particular, the Eqs. (1.1) and (1.2) can be used to describe the deflection of the beam [24, 35]. Let u and f respectively denote the deformation of a beam and the load density. Under small-deformation assumption, the u and f satisfy the equation

$$EI\Delta^2 u + T\Delta u = f,$$

where E, I and T are physical quantities. Different types of boundary conditions represent different beams end constraints — e.g.

- Under the pin-end boundary condition $\Delta u = u = 0$, the deformation is prescribed to be zero and there is no restoring moment on the boundary.
- Under built-in end boundary condition $\partial u/\partial \mathbf{n} = u = 0$, the deformation and the slope of the beam is zero on the boundary.

It is worth noting that many singular perturbation equations have a multi-scale solution. More precisely, the corresponding solution rapidly changes in boundary/interior layers and slowly outside of them. This effect generates numerous problems in numerical methods for such a kind of parameter-dependent equations. In order to achieve the required accuracy, the step size of the numerical scheme has to be much smaller than the parameter ϵ. Besides, the error bounds can blow up when parameter $\epsilon \to 0$. Therefore, numerical analysts are particularly interested in uniformly convergent numerical methods working for all possible parameters.

Uniformly convergent schemes for second-order SPEs are extensively studied during the last four decades [2, 5, 25, 34], but only a few research has focused on high order SPEs. Thus Liu [28] applied Hermitian scheme on a special non-equidistant mesh to fourth order singular perturbation problems with pin-end boundary conditions (FSPE-P) in 1-D situation and proved its fourth order convergence, uniform with respect to the small parameter ϵ. Shanthi and Ramanujam [36, 38, 39] transformed a 1-D FSPE-P into a second-order ODE system and solved it by fitted operator and fitted-mesh methods. They also proved that the errors of fitted operator and fitted-mesh methods are $\mathscr{O}(h + \sqrt{\epsilon})$ and $\mathscr{O}(h \ln(1/h) + \sqrt{\epsilon})$, respectively. For fourth-order singular perturbation problems with built-in end boundary condition (FSPE-B), Akram and Naheed [1] proposed a fourth order convergent scheme based on septic splines. Wang *et al.* [41] employed a modified Morley element method to FSPE-B and proved the uniform convergence of numerical solutions with respect to ϵ. Franz and Roos [7] constructed layer-adapted meshes for 2-D FSPE-B and established convergence rates in energy and balance norms.

The approximation methods mentioned play an important role in SPE studies. On the other hand, there are numerous works, which deal with the asymptotic analysis for SPEs — cf. [19, 24, 27, 29]. For example, applying a matched asymptotic method to the Eqs. (1.1) and (1.2), O'Malley [32] established the relation

$$u(x) = A_0(x) + \epsilon\left[B_0(x)e^{-x/\epsilon} + C_0(x)e^{-(1-x)/\epsilon}\right].$$