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Jacobi Spectral Galerkin Methods for a Class of Nonlinear Weakly Singular Volterra Integral Equations

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Abstract. We propose the Jacobi spectral Galerkin and Jacobi spectral multi Galerkin methods with their iterated versions for obtaining the superconvergence results of a general class of nonlinear Volterra integral equations with a kernel $x^{\beta}(z-x)^{-\kappa}$, where $0<\kappa<1$, $\beta>0$, which have an Abel-type and an endpoint singularity. The exact solutions for these types of integral equations are singular at the initial point of integration. First, we apply a transformation of independent variables to find a new integral equation with a sufficiently smooth solution. Then we discuss the superconvergence rates for the transformed equation in both uniform and weighted L^2- norms. We obtain the order of convergence in Jacobi spectral Galerkin method $\mathcal{O}(N^{\frac{3}{4}-r})$ and $\mathcal{O}(N^{-r})$ in uniform and weighted L^2- norms, respectively. Whereas iterated Jacobi spectral Galerkin method converges with the order of convergence $\mathcal{O}(N^{-2r})$ in both uniform and weighted L^2- norms. We also show that iterated Jacobi spectral multi Galerkin method converges with the orders $\mathcal{O}(N^{-3r}\log N)$ and $\mathcal{O}(N^{-3r})$ in uniform and weighted L^2- norms, respectively. Theoretical results are verified by numerical illustrations.

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Key words: Volterra integral equations with weakly singular kernels, Jacobi polynomials, spectral Galerkin method, spectral multi-Galerkin method, superconvergence results.

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1 Introduction

In this work, we discuss the numerical techniques for the approximations of the following Volterra integral equations with singularities

$$Y(\xi)=F(\xi)+\int_0^{\xi}\frac{x^{\beta}}{(x-\xi)^{\kappa}}g(Y(x))dx,\quad 0<\kappa<1,\quad \beta>0,\quad \xi\in[0,T], \tag{1.1}$$

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where the functions F and g are known and Y is the function to be approximated in a Banach space $\mathcal{X}$. These integral equations arise in many applications in science and technology such as mathematical physics and chemical reaction including chemical kinetics, heat conduction, theory of superfluidity, boundary layer and heat transfer [4, 5, 14, 16]. Note that the kernel of the integral equation (1.1),

$$k(x,\xi)=x^{\beta}(x-\xi)^{-\kappa}, \quad \kappa\in(0,1),$$

and $\beta>0$ has singularity along the diagonal $x=\xi$ and its first derivative has singularity at $x=0$ (depending on β). Since these types of integral equations can not be solved exactly in general, some numerical approximation methods are needed to find the approximate solutions. Some of the commonly known numerical techniques for obtaining the approximate solutions for the integral equations with weakly singular kernels are well documented in [2, 3, 6–11, 17, 19, 20]. Since the solution is nonsmooth at the initial point of integration, it is not easy to obtain the convergence rates in the uniform norm. Some approximation methods based on the graded mesh can be used to obtain the convergence results in the uniform norm. In [9], T. Diogo et al. discussed the fully discrete collocation method on graded mesh for nonlinear singular Volterra integral equations. In [17], M. Rebelo et al. discussed the hybrid collocation method which combines a non-polynomial approximation on the first subinterval followed by a piecewise polynomial collocation with graded meshes and calculated the order $\mathcal{O}(N^{-r})$. In [8], the extrapolation method for a nonlinear weakly singular Volterra integral equations has been discussed. In [3], P. Baratella discussed the Nystrom type interpolant of the solution based on Gauss Radau nodes for weakly singular Volterra-Hammerstein integral equations and used some smoothing transformations to improve the order of convergence in Nystrom methods. In [12], K. Kant et al. proposed the Galerkin method based on graded mesh for weakly singular Volterra-Hammerstein integral equations and discussed the convergence results. In the last decade, many authors have studied the spectral methods for Volterra integral equations. In [19], Ziqing Xie et al. considered the Jacobi spectral Galerkin method for linear Volterra integral equation and found the order $\mathcal{O}(N^{\frac{3}{4}-r})$ in the uniform norm and $\mathcal{O}(N^{-r})$ in the weighted L^2-norm, N and r denote the highest degree of the Jacobi polynomial employed in the approximation and smoothness of the solution, respectively. In [20], Yin Yan et al. discussed the convergence analysis of Legendre collocation methods for nonlinear Volterra type integral equations and obtained the order of convergence $\mathcal{O}(N^{\frac{1}{2}-r})$ in the uniform norm and $\mathcal{O}(N^{-r})$ in the weighted L^2-norm. In [6,7], Jacobi spectral methods have been discussed for weakly singular Volterra integral equations with smooth solutions. In [13], Li and Tang discussed the Jacobi spectral collocation method for a particular case of Abel-Volterra integral equation, i.e., with singular kernel $(x-\xi)^{-1/2}$, where the nonsmooth solution is considered. In [10] K. Kant et al., the Jacobi spectral Galerkin and multi Galerkin methods are considered for weakly singular Volterra-Urysohn integral equations and obtained the convergence analysis.

In [2], S. S. Allaei et al. proposed the Jacobi spectral collocation method for integral equations of the type (1.1) and obtained the order of convergence. Our main motivation