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The Plane Wave Methods for the Time-Harmonic Elastic Wave Problems with the Complex Valued Coefficients

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Abstract. In this paper the plane wave methods are discussed for solving the time-harmonic elastic wave propagation problems with the complex valued coefficients in two and three space dimensions. The plane wave least-squares method and the ultra-weak variational formulation are developed for the elastic wave propagation. The error estimates of the approximation solutions generated by the PWLS method are derived. Moreover, Combined with local spectral elements, the plane wave methods are generalized to solve the nonhomogeneous elastic wave problems. Numerical results verify the validity of the theoretical results and indicate that the resulting approximate solution generated by the PWLS method is generally more accurate than that generated by a new variant of the ultra-weak variational formulation method when the Lamé constants λ and μ are complex valued.

AMS subject classifications: 65N30, 65N55

Key words: Elastic waves, nonhomogeneous, plane wave least-squares, ultra-weak variational formulation, plane wave basis functions, error estimates, local spectral elements, preconditioner.

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1 Introduction

The plane wave method, which falls into the class of Trefftz methods [23], differs from the boundary-element method (BEM) and the traditional finite-element method (FEM) in the sense that the basis functions are chosen as exact solutions of the governing differential equation without boundary conditions. The plane wave method was first introduced to solve Helmholtz equations and was then extended to solve the Maxwell equations and time-harmonic elastic wave problems. Examples of this approach include the variational theory of complex rays (VTCR) [21], the ultra weak variational formulation (UWVF) [2,

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16, 26], the plane wave discontinuous Galerkin (PWDG) method [7–9, 25] and the plane wave least-squares (PWLS) method [12–14, 19, 20, 24].

The UWVF method was developed for the Helmholtz equations [2] and for Maxwell's equations [2]. The UWVF method is derived from non-overlapping domain decomposition with mixed interface conditions. The VTCR method developed in [21] is derived from an intuitive idea of imposing some weak continuity of the interface traces. In contrast to the VTCR method, the PWDG method developed in [7, 8] was derived from standard discontinuous Galerkin (DG) methods. We see that the choice $\alpha=\beta=\delta=1/2$ of flux parameters gives rise to the original UWVF introduced in [2].

Recently, the UWVF method was extended to solve elastic wave problems in [16]. The study [16] were devoted to approximating the $S-$ and $P-$wave components of the analytic solution in a balanced way for the accuracy and stability in two dimensional case. For the UWVF method, the traction of the approximation solution on the boundaries of every elements are chosen as the unknowns, and the conjugation of each traction has to be defined by introducing an additional mappings. Fortunately, the displacement field on the skeleton of the mesh can be recovered by the unknowns. However, the approximation to the displacement filed defined in each element with small size can not directly calculated unless the approximations are postprocessed for the case of complex coefficients. In the recently published work [13], the PWLS method was extended to solve the Maxwell equations. The innovation of the PWLS method in [13] lies in that the Robin-type traces are employed on the local interfaces to match the Robin-type boundary conditions in the Maxwell equations and two relaxation parameters are introduced in the objective functional. The PWLS method can obtain the approximation to the electromagnetic field when dealing with real or complex coefficients.

The main objectives of this paper are to develop the plane wave methods for the time-harmonic elastic wave problmes with the complex valued coefficients in two and three dimensions, to develop the associated numerical algorithms and to derive the convergence results for the present mehtod. The main results obtained in this paper are concluded as the following:

(i) For the homogeneous elastic wave problmes, we present the PWLS method (3.9), and derive the associated error estimates, see Theorems 5.1, 5.2 and 5.3.

(ii) Combined with local spectral elements, the PWLS method is generalized to solve the nonhomogeneous elastic wave problems. Moreover, we derive error estimates of the approximate solutions generated by this method, see Theorem 6.2.

(iii) Motivated by the UWVF mthod developed in [16], we develop a new variant of the UWVF, which can directly obtain the approximations to the displacement filed by choosing different trial functions and test functions. For convenience, we call the resulting variational formulation as UWVFD. The same procedure can also be generalized to the PWDG method developed in [6–8].