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A Linearised Three-Point Combined Compact Difference Method with Weighted Approximation for Nonlinear Time Fractional Klein-Gordon Equations

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Abstract. A numerical method for nonlinear time fractional Klein-Gordon equations is studied. Discretising spatial and temporal variables by a combined compact difference and a weighted approximation, respectively, we develop a linearised method for the equations under consideration. It has at least sixth-order accuracy in space and second-order accuracy in time. Numerical examples demonstrate the effectiveness and accuracy of the method.

AMS subject classifications: 65M06, 65N06

Key words: Combined compact difference method, finite difference method, nonlinear time fractional Klein-Gordon equation.

1. Introduction

In the past few decades, fractional differential equations have attracted substantial attention because of their excellent ability to describe the memory and the hereditary properties of various processes in physics, hydrology, and economics [3, 26, 33]. Nevertheless, closed-form analytical solutions of such equations are rarely available, hence the development of efficient numerical methods became an important task. The list of the corresponding numerical methods studied in recent years includes finite difference, finite element and spectral methods — cf. Refs. [4, 10–12, 15, 18, 23, 27, 29–31, 38, 40, 44].

The Klein-Gordon equations belong to the fundamental evolution equations describing nonlinear phenomena in condensed matter physics, classical, relativistic and quantum mechanics. They find applications in the study of propagation in superconductors, motion of pendulum, and dislocations in crystals [19, 41]. Numerical methods for such equa-

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tions have been developed by various authors. In particular, El-Sayed and Kaya [14, 25] considered the Adomian decomposition method (ADM) and comprehensively analysed its convergence. Later on, Jafari *et al.* [24] proposed a homotopy method to solve nonlinear Klein-Gordon equations with quadratic terms. It is more effective than the ADM as numerical experiments show. Abuteen *et al.* [1] developed a fractional reduced differential transform method for nonlinear fractional Klein-Gordon equations. Some other numerical methods are discussed in [5, 35].

We also note numerous works devoted to time fractional Klein-Gordon equations. Thus Cui [7] proposed a fourth-order compact scheme for the one-dimensional sine-Gordon equation, Golmankhaneh *et al.* [19] applied the homotopy perturbation method to nonlinear fractional Klein-Gordon equations, Vong and Wang [36, 37] developed a compact scheme and investigated its stability and convergence, Lyu and Vong [32] proposed a linearised second-order scheme for nonlinear time fractional Klein-Gordon equations (NTFKGEs), introduced its corresponding compact format which can achieve the fourth-order accuracy in space and the second-order accuracy in time.

With many existing fourth-order accuracy methods for NTFKGEs, there are only a few highly accurate numerical approaches to NTFKGEs. One of the ways to improve the accuracy is to employ the combined compact differences (CCD) method to discretise the space variable. Considering fractional advection-diffusion equations, Gao and Sun [16, 17] established three-point CCD methods, which use global Hermitian polynomials with continuous first- and second-order derivatives. Such methods can achieve sixth-order accuracy in space. Li *et al.* [28] developed a spatial sixth-order alternating direction implicit method for two-dimensional cubic nonlinear Schrödinger equations. He and Pan [22] proposed a three-level linearly implicit CCD together with ADI for generalised nonlinear Schrödinger equations with variable coefficients in two and three dimensions. For a more detailed information on the CCD method the reader can consult Refs. [6, 8, 9, 21, 34, 42].

Let us note that the presence of a nonlinear term in NTFKGEs leads to various problems when applying conventional discrete methods to the equation in question. Therefore, we employ linearisation. The main advantage of this approach consists in the evaluation of the nonlinear term at the previous time level — cf. [20, 32, 39, 43]. The major idea of this work is the use of a sixth-order CCD method along with weighted approximations in order to construct a linearised high accuracy method for NTFKGEs.

The remainder of the paper is organised as follows. In Section 2, we introduce a linearised three-point CCD method with weighted approximation for NTFKGEs. This method can achieve second-order accuracy in time and at least sixth-order accuracy in space. Numerical experiments, carried out in Section 3, demonstrate the high accuracy of the method. Finally, some brief conclusions are given in Section 4.

2. CCD Method with Weighted Approximation

Consider the following NTFKGEs equation:

$$ {}_{0}^{C}\mathrm{D}_{t}^{\alpha} u(x,t) = u_{xx}(x,t) - f(u) + p(x,t), \quad x \in (x_l, x_r), \quad t \in (0,T], \tag{2.1} $$