Closed-Form Expression for the Exact Period of a Nonlinear Oscillator Typified by a Mass Attached to a Stretched Wire

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Abstract. The exact analytical expression of the period of a conservative nonlinear oscillator with a non-polynomial potential, is obtained. Such an oscillatory system corresponds to the transverse vibration of a particle attached to the center of a stretched elastic wire. The result is given in terms of elliptic functions and validates the approximate formulae derived from various approximation procedures as the harmonic balance method and the rational harmonic balance method usually implemented for solving such a nonlinear problem.

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1 Introduction

We consider a one-dimensional oscillator of motion equation,

$$\ddot{x} + x - \frac{\lambda x}{\sqrt{x^2 + 1}} = 0, \qquad 0 \le \lambda \le 1. \tag{1.1}$$

This nonlinear ODE describes in a dimensionless form, for instance, the transverse vibration of a particle attached to the centre of a stretched elastic wire. Such an oscillator has been first analyzed by Mickens [1], and more recently reconsidered, among many others, by Beléndez et al. [2], Sun et al. [3] and Gimeno et al. [4]. The parameter λ is a geometrical deformation characteristic of the wire when the oscillator is at rest. For such a conservative system, the total energy

$$E = \frac{\dot{x}^2}{2} + \frac{x^2}{2} - \lambda\sqrt{x^2 + 1},$$

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is conserved and the motion of the particle is oscillatory for any energy value $E > -\lambda$. Notice that the energy is, in principle, defined up to a constant: for simplifying further the calculations, our convention is to fix the energy minimum at the value $-\lambda$, and not zero as customarily assumed in the works previously indicated.

It is known that the oscillation period depends on energy and is given by the integral [5],

$$T_\lambda(E) = \sqrt{2} \int_{-x_m(E)}^{x_m(E)} \left(E - \frac{1}{2}x^2 + \lambda \sqrt{x^2+1} \right)^{-\frac{1}{2}} dx, \tag{1.2}$$

where the turning points $-x_m$ and x_m $(x_m > 0)$ of the trajectory at energy E are the two real solutions of the equation,

$$E - \frac{1}{2}x^2 + \lambda\sqrt{x^2+1} = 0,$$

such that,

$$x_m = \sqrt{2}\sqrt{\sqrt{\lambda^2(\lambda^2 + 2E + 1)} + (\lambda^2 + E)}. \tag{1.3}$$

In addition, it is worth noting that if $\lambda \geq 1/2$ and $-\lambda < E \leq -1/2$, the latter equation has also the two following imaginary solutions,

$$\pm i\kappa = \pm i\sqrt{2}\sqrt{\sqrt{\lambda^2(\lambda^2 + 2E + 1)} - (\lambda^2 + E)}, \tag{1.4}$$

such that $0 < \kappa \leq 1$ (equality when $E = -1/2$).

The calculation of Eq. (1.2) is not straightforward, and to the best of our knowledge, such an exact result cannot be found anywhere. Only approximate solutions exist which are derived, for instance, by means of the harmonic balance method [2,3], the rational balance method [4], the energy balance method [6], the parameter-expansion method [7] or using a variational approach [8] when applied for solving Eq. (1.1).

In this note, we derive an exact closed-form expression for the period function $T_\lambda(E)$ in terms of complete elliptic integrals of the first and third kinds. Such a result is obtained thanks to appropriate integrations in the complex plane, as detailed in Sections 2 and 3. We summarize our main result in Section 4 and give, finally in the last section, accurate estimates of the period for small and large amplitude oscillations which validate the nonlinear analytical methods usually employed for solving this problem.

2 Integration in the complex plane

2.1 case I : for $0 < \lambda < \frac{1}{2}$

The function of a complex variable $z = x + iy$,

$$f(z) = \left(E - \frac{1}{2}z^2 + \lambda\sqrt{z^2+1} \right)^{-\frac{1}{2}}, \tag{2.1}$$