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Completion of $\mathbb{R}^2$ with a Conformal Metric as a Closed Surface

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper, we obtain some asymptotic behavior results for solutions to the prescribed Gaussian curvature equation. Moreover, we prove that under a conformal metric in $\mathbb{R}^2$, if the total Gaussian curvature is 4π, the conformal area of $\mathbb{R}^2$ is finite and the Gaussian curvature is bounded, then $\mathbb{R}^2$ is a compact $C^{1,\alpha}$ surface after completion at ∞, for any $\alpha \in (0,1)$. If the Gaussian curvature has a Hölder decay at infinity, then the completed surface is C^2. For radial solutions, the same regularity holds if the Gaussian curvature has a limit at infinity.

Key Words: Gaussian curvature, conformal geometry, semilinear equations, entire solutions.

AMS Subject Classifications: 35B08, 35J15, 35J61, 53C18

1 Introduction

In this paper, we consider the prescribed Gaussian curvature equation

$$\Delta u + K(x)e^{2u} = 0 \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where K satisfies

$$\int_{\mathbb{R}^2} K(x)e^{2u(x)}\, dx < \infty. \tag{1.2}$$

(1.1) is equivalent to that K is the Gaussian curvature of $(\mathbb{R}^2, e^{2u}\delta)$, where δ is the Euclidean metric, and hence (1.2) means that the total Gaussian curvature is finite. A natural question is the following:

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Question 1.1. If u is an entire C^2 solution to (1.1), then by assuming what conditions on $K(x)$ can $(\mathbb{R}^2, e^{2u}\delta)$ be a C^2 closed Riemannian surface after completion at ∞?

Note that necessary conditions for this to be true include that

$$\int_{\mathbb{R}^2} K(x)e^{2u(x)}dx = 4\pi \quad \text{(Gauss-Bonnet Theorem)}, \tag{1.3a}$$

$$e^{2u} \in L^1(\mathbb{R}^2) \quad \text{(finite conformal area)}, \tag{1.3b}$$

$$\lim_{|x|\to\infty} K(x) = K_\infty \quad \text{uniformly, for some } K_\infty \in \mathbb{R}. \tag{1.3c}$$

A natural question is the following:

Question 1.2. Are (1.3a)-(1.3c) sufficient to guarantee that $(\mathbb{R}^2, e^{2u}\delta)$ is a C^2 closed Riemannian surface after completion at ∞?

This question is related to a more general question in [6] (Question 8.3) regarding the total area of $\mathbb{R}^2$ equipped with a conformal metric $e^{2u}\delta$ with its Gaussian curvature bigger than 1, i.e., with u being a super solution of (1.1).

Notice that (1.3c) implies that

$$K \in L^\infty(\mathbb{R}^2). \tag{1.4}$$

For the convenience of later discussion, we define

$$\lambda := \frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u(x)}dx.$$

In the following, (1.3b) and (1.4) will serve as fundamental assumptions.

Using the stereographic projection, we can identify conformally $\mathbb{R}^2$ with the unit sphere in $\mathbb{R}^3$ without the north pole. To complete the manifold, we need to find a coordinate system near the north pole so the metric is C^2 there. It is natural to use the Kelvin transform $x \mapsto \frac{x}{|x|^2}$ to convert the infinity of $\mathbb{R}^2$ to the origin when $\mathbb{R}^2$ is identified with the complex plane, and hence obtain the local coordinate system near the north pole. From analytic point of view, the completion of $(\mathbb{R}^2, e^{2u}\delta)$ is a closed C^2 Riemannian surface, if and only if

$$h(x) := e^{2u(\frac{x}{|x|^2})}\frac{1}{|x|^4} \tag{1.5}$$

is a C^2 function near $x = 0$, and $\lim_{|x|\to 0} h(x) > 0$, which means the metric at ∞ is nondegenerate. Hence we need to closely study the asymptotic behavior of u at ∞.

We are mostly interested in the case $\lambda = 2$ since it corresponds to (1.3a).

Our first result concerns the asymptotic behavior of u and its partial derivatives when $|x|$ is large.