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Riemann-Hilbert Approach and Soliton Solutions of the Higher-Order Dispersive Nonlinear Schrödinger Equations with Single and Double Poles

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Abstract. The higher-order dispersive nonlinear Schrödinger equation with the zero boundary conditions at the infinity is studied by the Riemann-Hilbert approach. We consider the direct scattering problem, corresponding eigenfunctions, scattering matrix and establish some of their properties. These results are used in the construction of an associated Riemann-Hilbert problem. Assuming that the scattering coefficients possess single or double poles, we derive the problem solutions. Finally, we present graphical examples of 1-, 2- and 3-soliton solutions and discuss their propagation.

AMS subject classifications: 35C08, 35Q15, 35Q51

Key words: Higher-order dispersive nonlinear Schrödinger equation, Riemann-Hilbert approach, soliton solutions.

1. Introduction

Nonlinear Schrödinger (NLS) equations are fundamental physical models arising in various fields, including deep water waves [39], plasma physics [40, 41], nonlinear optical fibers [2, 14]. The Kaup-Newell equation [15], the Chen-Lee-Liu equation [5] and the Gerdjikov-Ivanov equation [11, 26] are three well-known derivative NLS equations, which play an important role in mathematical physics. Since the gauge transformations translate these equations into each other, the gauge transformation method is widely used [9]. Other extended NLS equations have been also studied. In particular, various exact solutions of nonlinear evolution equations, such as lumps, rogue waves and travelling waves have been found [13, 33, 43]. The methods developed to obtain exact solutions include inverse scattering transformation [1, 4, 8, 16, 20, 21, 23, 27, 37] and Riemann-Hilbert approach [10, 12, 18, 19, 24, 25, 28, 29, 32, 34, 35, 38].

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Motivated by the works mentioned, we study the higher-order dispersive nonlinear Schrödinger (HDNLS) equation [22,36] with zero boundary conditions (ZBCs) at infinity. The HDNLS equation takes the form

$$iu_t + u_{xx} + 2|u|^2u + \tau\left(u_{xxxx} + 8|u|^2u_{xx} + 2u^2u^*_{xx} + 4u|u_x|^2 + 6u^*u_x^2 + 6|u|^4u\right) = 0, \quad (1.1)$$

where u is a complex function of variables x and t representing the slowly changing envelope of the wave, and $\tau = \epsilon^2/12$ a small dimensionless parameter. If $\tau = 0$, the Eq. (1.1) is the classical NLS equation. It is known that the HDNLS equation (1.1) can be used to control ultrashort optical pulse propagation in a long-distance high-speed fiber transmission systems with higher-order nonlinear effects such as fourth-order dispersion [3,7] and takes part in describing nonlinear one-dimensional isotropic biquadratic Heisenberg ferromagnetic spins with the octupole-dipole interaction [6,22]. It was studied by the Darboux transformation method [30,42], a modified Darboux transformation method [31], and the inverse scattering transform [17].

In this work, we employ the Riemann-Hilbert approach based on the inverse scattering transform to classify the solitons generated by the HDNLS equation with single and double poles. As far as we know, for the HDNLS equation the problem of single and double poles under zero boundary conditions has not been studied yet. The outline of the paper is as follows. In Section 2, we consider the direct scattering problem for the HDNLS equation with zero boundary conditions. Section 3 deals with the inverse scattering problem for the HDNLS equation and with the construction of the corresponding Riemann-Hilbert problem (RHP). The Riemann-Hilbert problem is solved for both cases of single and double poles and soliton solutions are described. In Section 4, we discuss some interesting phenomena by using graphic representations of special solutions. Section 5 contains our conclusions and discussions.

2. Direct Scattering Problem

We start with eigenfunctions and their analytic properties. According to [22], the Eq. (1.1) has the Lax pair

$$\psi_x = X\psi, \quad \psi_t = T\psi, \quad (2.1)$$

where $\psi \equiv \psi(x,t,k)$ is a 2×2 matrix function and

$$\begin{aligned}
X &= -ik\sigma_3 + U,\\
T &= \Big[3i\tau|u|^4 + i|u|^2 + i\tau\left(u^*u_{xx} + uu^*_{xx} - |u_x|^2\right)\\
&\qquad + 8i\tau k^4 + 2k\tau(uu^*_x - u_xu^*) - 2ik^2(2\tau|u|^2+1)\Big]\sigma_3\\
&\quad - 8\tau k^3U - 4i\tau k^2\sigma_3U_x + 6i\tau U^2U_x\sigma_3\\
&\quad + i\sigma_3U_x + i\tau\sigma_3U_{xxx} + 2k\left(U + \tau U_{xx} - 2\tau U^3\right)
\end{aligned}$$

with

$$U = \begin{pmatrix} 0 & u(x,t) \\ -u^*(x,t) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$