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An Efficient Iterative Approach to Large Sparse Nonlinear Systems with Non-Hermitian Jacobian Matrices

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Abstract. Inner-outer iterative methods for large sparse non-Hermitian nonlinear systems are considered. Using the ideas of modified generalised Hermitian and skew Hermitian methods and double-parameter GHSS method, we develop a double-parameter modified generalised Hermitian and skew Hermitian method (DMGHSS) for linear non-Hermitian systems. Using this method as the inner iterations and the modified Newton method as the outer iterations, we introduce modified Newton-DMGHSS methods for large sparse non-Hermitian nonlinear systems. The convergence of the methods is studied. Numerical results demonstrate the efficacy of the methods.

AMS subject classifications: 65F10, 65F50, 65H10

Key words: Splitting iteration, positive definite Jacobian matrices, large sparse nonlinear system, modified Newton-DMGHSS method, convergence.

1. Introduction

Numerical solution of nonlinear partial differential equations — e.g. the Poisson-Boltzmann equation, are vital to many scientific and engineering applications. The discretisation of such equations leads to a nonlinear system

$$F(x) = 0 \tag{1.1}$$

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with a continuously differentiable (nonlinear) map F defined on a domain in $\mathbb{R}^n$ or $\mathbb{C}^n$. It is worth noting that the systems arising are often connected with large sparse matrices — cf. [19, 20]. This study focuses on numerical solutions x of nonlinear systems (1.1) with a sparse non-Hermitian, positive definite Jacobian matrix $F'(x)$.

Inexact Newton approaches [15], which do not use the inverses of the Jacobian matrices, are an attractive tool for the solution of such systems. This method can be described as follows.

Algorithm 1.1 Inexact Newton Method.

1: Give an initial guess x_0.
2: **for** $k = 0$ **to** "convergence" **do**
3: Develop some $\eta_k \in [0, 1)$ and s_k that meet $\|F(x_k) + F'(x_k)s_k\| \leq \eta_k \|F(x_k)\|$.
4: **end for**
5: Set $x_{k+1} = x_k + s_k$.

Here, $F'(x_k)$ is the Jacobian matrix and $\eta_k \in [0, 1)$ a forcing term used to control the accuracy. An inexact Newton approach consists of two steps. The nonlinear iteration generating a sequence $\{x_k\}$ is referred to as the outer iteration and the linear iteration generating an approximation for the Newton step s_k as the inner iteration. We note that linear iterative methods often used as internal Newton iterative solvers lead to inner-outer iterations such as Newton-CG and Newton-GMRES — cf. [3, 17]. In particular, using a Hermitian and skew-Hermitian splitting method [6] in the inexact Newton procedure, one obtains the so-called Newton-HSS method — cf. [10]. It is widely used for solving large sparse nonlinear systems with non-Hermitian positive definite Jacobian matrices.

This work is aimed at the development of efficient and robust iteration methods, which have a high convergence order, for nonlinear systems. For example, the modified Newton method

$$\begin{cases} y_k = x_k - F'(x_k)^{-1} F(x_k), \\ x_{k+1} = y_k - F'(x_k)^{-1} F(y_k), \quad k = 0, 1, 2, \dots \end{cases} \tag{1.2}$$

has at least the third order of convergence [14]. It is comparable with the Newton method, and it has been widely used in for solving inverse eigenvalue problems [13], power flow equations [16], and other nonlinear systems [21, 22]. Wu and Chen [21] considered a modified Newton-HSS method based on a modified Newton method and the Hermitian and skew-Hermitian splitting as outer and inner solvers, respectively. Note that the convergence analysis shows that the efficiency of the inner iteration affects the convergence rate of the modified Newton-HSS approach [12]. Therefore, at each Newton iteration an efficient inexact solver for the Jacobian systems has to be employed.

For linear system $Ax = b$, the Hermitian and skew-Hermitian splitting (HSS) [6] and its modifications [4, 5, 7–9, 11] are popular methods because of their efficiency. Using the HSS method, Benzi [11] developed a generalised Hermitian and skew-Hermitian splitting iterations (GHSS) for positive definite, non-Hermitian linear systems. It finds applications in Sylvester equations [23] and image restoration [1]. The GHSS method is based on