East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.190520.111020

Vol. **11**, No. 2, pp. 255-275
May 2021

Finite Difference Methods for Fractional Differential Equations on Non-Uniform Meshes

Haili Qiao and Aijie Cheng*

School of Mathematics, Shandong University, Jinan, Shandong 250100, China.

Received 19 May 2020; Accepted (in revised version) 11 October 2020.

Abstract. The solutions of fractional equations with Caputo derivative often have a singularity at the initial time. Therefore, for numerical methods on uniform meshes it is difficult to achieve optimal convergence rates. To improve the convergence, Liu *et al.* [10] considered a finite difference method on non-uniform meshes. Following the ideas of [10], we introduce two more sets of non-uniform meshes and show that the corresponding discrete models have higher convergence rates. Besides, we apply the trapezoidal rule in the case of linear fractional partial differential equations. The results of numerical experiments are consistent with the theoretical analysis.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Fractional differential equation, weak singularity, finite difference, convergence analysis, non-uniform meshes.

1. Introduction

Let $0 < \alpha < 1$, $\Omega = (a, b)$, and ${}_0^C D_t^\alpha$ denote the Caputo derivative — i.e.

$$ {}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-a)} \int_0^t \frac{u'(\tau)}{(t-\tau)^\alpha} d\tau. $$

In this work, we consider two equations with the Caputo fractional derivative — viz. the nonlinear ordinary differential equation

$$ \begin{aligned} &{}_0^C D_t^\alpha u(t) = f(t, u(t)), \quad t \in (0, T], \\ &u(0) = u_0, \end{aligned} \tag{1.1} $$

and the linear fractional partial differential equation

$$ \begin{aligned} &{}_0^C D_t^\alpha u(x,t) - p\frac{\partial^2 u}{\partial x^2}(x,t) + c(x)u(x,t) = f(x,t), && (x,t) \in \Omega \times (0,T], \\ &u(x,0) = 0, && x \in \Omega, \\ &u(x,t) = 0, && x \in \partial\Omega, \quad 0 \le t \le T, \end{aligned} \tag{1.2} $$

*Corresponding author. *Email addresses:* `QHLmath@163.com` (H. Qiao), `aijie@sdu.edu.cn` (A. Cheng)

where $p > 0$ is a positive constant and $c(x) \in C(\overline{\Omega})$ a non-negative function. If f is a continuous function satisfying the Lipschitz condition with respect to the second argument on a set G, Diethelm and Ford [4, Theorems 2.1 and 2.2] proved the unique solvability of the Eq. (1.1).

According to [4, Lemma 2.3], the Eq. (1.1) can be reduced to the following integral equation:

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, u(s)\big) ds. \tag{1.3}$$

Although analytic solutions of (1.1) can be rarely found, there are various numerical methods for its solution. Thus [1] considers an improved block-by-block method having a high convergence order for sufficiently smooth solutions; [6] reduces (1.1) to integral equation (1.3) and employs fractional Euler and Adams methods. The paper [8] also uses a high-order method. All the works mentioned assume that the solution of (1.1) is sufficiently smooth. However, the solution of the fractional order differential equations, very often have a weak singularity at the initial time and it is difficult to obtain optimal error estimates for the corresponding numerical schemes. Various methods, including introduction of correction terms [17], graded meshes [11] and non-uniform [10] meshes, have been proposed to improve the convergence. Assuming that ${}_0^C D_t^\alpha u(t)$ is not sufficiently smooth — cf. Assumption 1.1, Liu *et al.* [10] employed a finite difference discretisation on non-uniform meshes and obtained ideal convergence rates. Following the ideas of this work, we consider two more non-uniform meshes. Similar theoretical analysis shows that a higher convergence order can be obtained for the numerical methods on these meshes. Numerical experiments confirm the theoretical findings.

Assumption 1.1 (cf. Liu *et al.* [10])**.** Let $0 < \alpha < 1$, $0 < \sigma < 1$, and $g(t) :={}_0^C D_t^\alpha u(t)$. Then there is a constant $C > 0$ such that

$$|g'(t)| \le C t^{\sigma-1},$$

where $g'(t)$ denotes the first derivatives of $g(t)$.

The rationality of the hypothesis is explained in [10, 11].

2. Numerical Methods

In this section, we consider approximation methods for the Eq. (1.3), which use finite difference on three non-uniform meshes. The integral term is approximated by rectangular and trapezoidal formulas — cf. [10]. Since the numerical expressions obtained by using trapezoidal formula for nonlinear equations is relatively complicated, Liu *et al.* [10] introduced a prediction correction method.

For a positive integers N, let $0 = t_0 < t_1 < \cdots < t_N = T$ be the corresponding non-uniform meshes. We consider three discrete schemes — viz.