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A Linearized Difference Scheme for Time-Fractional Sine-Gordon Equation

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Abstract. In this paper, a linearized difference scheme is proposed for the Sine-Gordon equation (SGE) with a Caputo time derivative of order $\alpha\in(1,2)$. Comparing with the existing linearized difference schemes, the proposed numerical scheme is simpler and easier for theoretical analysis. The solvability, boundedness and convergence of the difference scheme are rigorously established in the L_∞ norm. Finally, several numerical experiments are provided to support the theoretical results.

AMS subject classifications: 35R11, 65M06, 65M12

Key words: Time-fractional Sine-Gordon equation, Caputo fractional derivative, linearized difference scheme, convergence and stability.

1 Introduction

The nonlinear SGE

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+\sin u=0, \quad \mathbf{x}=(x_1,\cdots,x_n)\in\mathbb{R}^n,$$

is an important equation in the description of many physical phenomenon, such as Josephson junctions, the propagation of fluxon, stability of fluid motions, models of particle physics [1,2], etc.

Due to the temporal memory or nonlocal properties of fractional calculus, fractional differential equations are always better than corresponding integer-order differential equations in the description of some phenomena in the real world. Therefore various fractional order differential equations have attract many researchers' attention [3–15].

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But there are few researches on fractional order SGE. In [16], the author found the supratransmission phenomenon present in Riesz space-fractional sine-Gordon systems. By applying the rotating wave approximation, a family of breather-like solutions for Riesz space-fractional SGE is found numerically [17]. Ray [18] combined the modified decomposition method and Fourier transform to approximate solution of fractional SGE. In [19], we proposed a conservative difference scheme for the Riesz space-fractional SGE, and introduced a revised Newton method to reduce the computational complexity of performing the numerical scheme.

In this paper, we consider the time-fractional SGE

$$ {}_0^C D_t^\alpha u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + \sin u(x,t) = f(x,t), \quad x\in(a,b), \quad t\in(0,T], \tag{1.1} $$

subject to the initial conditions

$$ u(x,0)=\varphi_1(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi_1(x), \quad x\in(a,b), \tag{1.2} $$

and the boundary conditions

$$ u(a,t)=u(b,t)=0, \quad 0<t\le T, \tag{1.3} $$

where

$$ {}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}} $$

is the Caputo fractional derivative of order $\alpha\in(1,2)$.

If directly apply the difference method proposed in [20] to discretize the Caputo fractional derivative ${}_0^C D_t^\alpha u(x,t_{n-1/2})$, and approximate the nonlinear term $\sin u(x,t_{n-1/2})$ by $(\sin u(x,t_n)+\sin u(x,t_{n-1}))/2$, it will lead to a nonlinear difference equation. What's worse, due to iterative method is needed to solve the obtained nonlinear equations, the implementation of the numerical scheme will require a large computational cost. From many research works, we see that linearized schemes are very efficient in dealing with nonlinear differential equations [3,22–26]. In [26], a linearized difference scheme was proposed to solve nonlinear time fractional Klein-Gordon equations. The difference scheme is second-order in time. However, the theoretical analysis of the numerical scheme is rather complicated. In this paper, we propose a linearized difference scheme, which is inspired by the numerical scheme proposed in [20] and the idea of linearized method. Compared with the numerical scheme given in [26], the proposed numerical scheme is simpler and easier to analyze theoretically.

This paper is arranged as follows. In the next section, a linearized difference scheme is proposed for the time-fractional SGE. Subsequently, the solvability, boundedness and convergence of the difference scheme are rigorously established. In Section 4, we present some numerical results to demonstrate the effectiveness of the difference scheme. Finally, we give a simple conclusion.