Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 4 (2020), pp. 468-481

DOI: 10.4208/ata.OA-SU4

A Note on $Card(X)$

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Received 17 September 2020; Accepted (in revised version) 21 September 2020

Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. The main interests here are to study the relationship between $card(X)$ and $card(\mathcal{P}(X))$ and the connection between the separability of a space X and cardinality of some function space on it. We will convert the calculation of $card(\mathcal{P}(X))$ to the calculation of $card(\mathcal{F}(X \to \mathbb{Q}))$. The main tool we used here is Zorn Lemma.

Key Words: Cardinality, separability of space, Zorn Lemma.

AMS Subject Classifications: 03E10

1 Introduction

Let X be a set. If X is a finite set, we call the number of elements of X the cardinality of X, and denote it by $card(X)$. For two infinite sets X and Y, we can use this notion to compare the "number" of two sets X and Y. The following expressions are well-known:

(i) $card(X) \leq card(Y)$ if there exists an injective map $\phi : X \to Y$;

(ii) $card(X) \geq card(Y)$ if there exists a surjective map $\phi : X \to Y$;

(iii) $card(X) = card(Y)$ if there exists a bijective map $\phi : X \to Y$.

Let X and Y be two sets. We recall the following theorems in [1–3].

Theorem 1.1. *$card(X) = card(Y)$ if and only if $card(X) \leq card(Y)$ and $card(X) \geq card(Y)$ both hold.*

Theorem 1.2. *Either $card(X) < card(Y)$ or $card(Y) < card(X)$ or $card(X) = card(Y)$.*

Theorem 1.3. *$card(X) < card(\mathcal{P}(X))$.*

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In this paper, we use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the set of positive integers, integers, rational numbers, real numbers and complex numbers respectively. The number filed F mentioned here is a subfield of $\mathbb{C}$, thus $\mathbb{Q}$ is the minimal number field and $F \supset \mathbb{Q}$. Given two sets X and Y, we denote

$$\mathcal{F}(X \to Y) = \{\text{map } f : X \to Y\}. \tag{1.1}$$

Especially, there is a natural algebra structure on $\mathcal{F}(X \to F)$ if F is a field. As usual, we use (X,ρ) to denote a metric space with a metric map $\rho : X \times X \to [0,+\infty)$, which satisfies

(i) $\rho(x_1,x_2) = 0$ if and only if $x_1 = x_2$;

(ii) $\rho(x_1,x_2) = \rho(x_2,x_1)$;

(iii) $\rho(x_1,x_3) \le \rho(x_1,x_2) + \rho(x_2,x_3)$, where x_1,x_2,x_3 are arbitrary points of X.

We use $(X,\mathcal{M},\mu)$ to denote a measure space, where $\mathcal{M}$ is a σ-algebra on X, and μ is a measure, i.e., $\mu : \mathcal{M} \to [0,+\infty]$ is a map, satisfying

(i) $\mu(\phi) = 0$;

(ii) $\mu(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{+\infty} \mu(E_j)$, where $E_j \in \mathcal{M}$ and $E_{j_1} \cap E_{j_2} = \emptyset$, $(j_1 \neq j_2)$.

We denote $card(\mathbb{N}) = c_0$, which is the minimal cardinality of all infinite sets. Denote $card(\mathbb{R}) = c$, which is called "cardinality of the continuum".

Let X and Y be two sets and $\alpha = card(X)$, $\beta = card(Y)$. We have the following definitions,

Definition 1.1. If $X \cap Y = \emptyset$, we define $\alpha + \beta = card(X \cup Y)$.

Definition 1.2. Define $\alpha \cdot \beta = card(X \times Y)$.

Definition 1.3. Define $\beta^{\alpha} = card(\mathcal{F}(X \to Y))$.

We verify that these three definitions are well-defined. Suppose two sets X_1 and Y_1 satisfy $card(X_1) = card(X)$, $card(Y_1) = card(Y)$ and $X_1 \cap Y_1 = \emptyset$ (in Definition 1.1). Then, we have bijective maps $\phi : X \to X_1$ and $\psi : Y \to Y_1$. We construct three maps ω, θ, η as follows:

$$\omega : X \cup Y \to X_1 \cup Y_1, \quad \omega(z) = \begin{cases} \phi(x), & \text{if } z = x \in X, \\ \psi(y), & \text{if } z = y \in Y, \end{cases} \tag{1.2a}$$

$$\theta : X \times Y \to X_1 \times Y_1 : \ \theta(x,y) = (\phi(x),\psi(y)), \tag{1.2b}$$

where $x \in X, y \in Y$.

$$\eta : \mathcal{F}(X \to Y) \to \mathcal{F}(X_1 \to Y_1) : \ \eta(f) = \psi \circ f \circ \phi^{-1}, \tag{1.3}$$

where $f \in \mathcal{F}(X \to Y)$, "$\circ$" represents the composition of maps. It is easy to verify that ω,θ,η are bijective. Thus these definitions are well-defined.