East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.140820.250820

Vol. **11**, No. 1, pp. 119-142
February 2021

# Regularised Finite Difference Methods for the Logarithmic Klein-Gordon Equation

Jingye Yan[1], Hong Zhang[1], Xu Qian[1,*] and Songhe Song[1,2]

[1]*Department of Mathematics, College of Arts and Sciences, National University of Defense Technology, Changsha 410073, China.*
[2]*State Key Laboratory of High Performance Computing, National University of Defense Technology, Changsha 410073, China.*

*Received 14 August 2020; Accepted (in revised version) 25 August 2020.*

**Abstract.** Two regularised finite difference methods for the logarithmic Klein-Gordon equation are studied. In order to deal with the origin singularity, we employ regularised logarithmic Klein-Gordon equations with a regularisation parameter $0 < \varepsilon \ll 1$. Two finite difference methods are applied to the regularised equations. It is proven that the methods have the second order of accuracy both in space and time. Numerical experiments show that the solutions of the regularised equations converge to the solution of the initial equation as $\mathscr{O}(\varepsilon)$.

**AMS subject classifications**: 35Q40, 65M15, 81Q05

**Key words**: Logarithmic Klein-Gordon equation, regularised logarithmic Klein-Gordon equation, finite difference method, error estimate, convergence order.

## 1. Introduction

The logarithmic Klein-Gordon equation (LogKGE), also known as the relativistic version of the logarithmic Schrödinger equation [15], has been introduced in the quantum field theory by Rosen [41]. It has the form

$$\begin{aligned} &u_{tt}(\mathbf{x},t)-\Delta u(\mathbf{x},t)+u(\mathbf{x},t)+\lambda u(\mathbf{x},t)\ln\left(|u(\mathbf{x},t)|^2\right)=0, && \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ &u(\mathbf{x},0)=\phi(\mathbf{x}), \quad \partial_t u(\mathbf{x},0)=\gamma(\mathbf{x}), && \mathbf{x}\in\mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\mathbf{x}=(x_1,\dots,x_d)^{\mathrm{T}}\in\mathbb{R}^d$, $d=1,2,3$ is the spatial coordinate, $t$ time, $u:=u(\mathbf{x},t)$ a real-valued scalar field, and $\lambda$ shows the force of the non-linear interaction. Such non-linearities appear in relativistic wave equations, which describe dilatonic quantum gravity [43], superfluid [44], spinless particles [16, 17] and non-relativistic spinning particles moving in an external electromagnetic field. Besides, such non-linearity effects often arise in various

*Corresponding author. *Email addresses:* yanjingye0205@163.com (J. Yan), zhanghnudt@163.com (H. Zhang), qianxu@nudt.edu.cn (X. Qian), shsong@nudt.edu.cn (S. Song)

 

areas of physics such as inflation cosmology [14, 25], supersymmetric field theories, geophysics [32, 37], optics [20], and nuclear physics [34]. The LogKGE (1.1) has been also used to describe spinless particles in optics [42]. If $u(\cdot,t) \in H^1(\mathbb{R}^d)$ and $\partial_t u(\cdot,t) \in L^2(\mathbb{R}^d)$, then the LogKGE (1.1) admits the energy conservation law — cf. [35, 38], i.e.

$$\begin{aligned} E(t) &= \int_\Omega \Big[ |u_t(\mathbf{x},t)|^2 + |\nabla u(\mathbf{x},t)|^2 + (1-\lambda)|u(\mathbf{x},t)|^2 + \lambda |u(\mathbf{x},t)|^2 \ln\big(|u(\mathbf{x},t)|^2\big)\Big] dx \\ &\equiv E(0). \end{aligned}$$

The global-in-time well-posedness of the solution to the Klein-Gordon equation with a logarithmic potentials attracted considerable attention. Thus Cazenave and Haraux [22] studied the local existence and uniqueness of solution of the Cauchy problem. Later on, Górka [27] used the compactness method to show the global existence of weak solutions for one-dimensional equations in bounded domains. Bartkowski and Górka [15] studied the corresponding Cauchy problem on the real line $\mathbb{R}$ without boundary conditions. They established the global existence of weak solutions, classical solutions and traveling waves. Natali and Cardoso Jr. [39] used compactness arguments and a non-standard analysis to prove the existence and uniqueness of weak solutions for an associated Cauchy problem in the energy space. Bialynicki-Birula and Mycielski [18] studied the Gaussons — i.e. the solutions, which represent the Gaussian shape [48]. Note that the interaction of Gaussons was first considered by Makhankov *et al.* [36]. For the nonlinear Klein-Gordon equation (NKGE) and the oscillatory NKGE, Cauchy problems, well-posedness and dynamical properties are investigated in [1, 19, 29, 31, 45].

Numerical methods have for the non-linear Klein-Gordon equation (NKGE) and the oscillatory NKGE have been also developed. Thus standard finite difference time domain (FDTD) methods such as energy conservative, semi-implicit, explicit finite difference time domain are considered in [8, 10, 23, 24, 47, 50], a multiscale time integrator Fourier pseudospectral (MTIFP) method in [4, 11, 12], a finite element method in [21], an exponential wave integrator Fourier pseudospectral (EWI-FP) method in [8, 9], and an asymptotic preserving (AP) method in [26]. For numerical comparison of various numerical methods for the NKGE and the oscillatory NKGE, we refer the reader to [8, 13, 30, 40]. Stochastic conformal Preissman, stochastic conformal discrete gradient and stochastic conformal Euler box schemes for damped stochastic Klein-Gordon equation are presented in [46]. Nevertheless, these methods can not be directly applied to the LogKGE equation (1.1) because of the singularity of the logarithmic non-linearity at the origin.

Considering the logarithmic Schrödinger equation (LogSE), Bao *et al.* [6] employed regularised mass and energy conservative finite difference methods in order to avoid the blowup of the logarithmic non-linearity. Latter on, a regularised semi-implicit difference scheme for the LogSE was studied in [5]. Li *et al.* [33] proposed a Crank-Nicolson-type finite difference method for the LogSE on unbounded domains. Zhang *et al.* [49] developed high order diagonally Runge-Kutta schemes for the LogSE, which preserve the mass and quadratic energy and introduced multi-symplectic integrators which conserve mass. In order to avoid the singularity of the logarithmic function, Bao *et al.* [7] applied Lie-Trotter