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An Energy-Preserving Scheme for the Coupled Gross-Pitaevskii Equations

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Abstract. An energy-preserving scheme is proposed for the coupled Gross-Pitaevskii equations. The scheme is constructed by high order compact method in the spatial direction and average vector field method in the temporal direction, respectively. The scheme is energy-preserving, stable, and of sixth order in space and of second order in time. Numerical experiments verify the theoretical results. The dynamic behavior modeled by the coupled Gross-Pitaevskii equations is also numerically investigated.

AMS subject classifications: 65M06, 65M12, 65Z05, 70H15

Key words: Coupled Gross-Pitaevskii equations, average vector field method, high order compact method, energy-preserving scheme.

1 Introduction

In Bose-Einstein condensates (BECs) community, using mean field approximation, the two-component time-dependent dimensionless coupled Gross-Pitaevskii (CGP) equations with external potentials and effective Rabi frequencies is as follows [11, 16, 22, 23, 25, 35]

$$iu_t = \Big(-\frac{1}{2}\Delta + s_1(x,y) + \beta_{11}|u|^2 + \beta_{12}|v|^2\Big)u - L_z u - \lambda v, \qquad (x,y)\in\mathbb{R}^2, \qquad t>0, \tag{1.1a}$$

$$iv_t = \Big(-\frac{1}{2}\Delta + s_2(x,y) + \beta_{21}|u|^2 + \beta_{22}|v|^2\Big)v - L_z v - \lambda u, \qquad (x,y)\in\mathbb{R}^2, \qquad t>0, \tag{1.1b}$$

where $i^2=-1$, and Δ is the Laplace operator. We only consider two-dimensional case, i.e., $\Delta=\partial_{xx}+\partial_{yy}$ in this paper. $u(x,y,t)$, $v(x,y,t)$ are the complex-valued wave functions.

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$s_l(x,y)$ ($l=1,2$) are the real-valued external trapping potential functions. In the harmonic potentials, they are generally in the forms

$$s_l(x,y)=\frac{1}{2}(\omega_{x,l}^2x^2+\omega_{y,l}^2y^2)$$

with dimensionless frequencies $\omega_{x,l}$ and $\omega_{y,l}$ ($l=1,2$) in the x and y directions, respectively. The action between intra-component and inter-component atomic is represented by

$$\beta_{lk}=\beta_{kl}=\frac{4\pi N_0\alpha_{lk}}{\alpha_0},\quad k,l=1,2,$$

with N_0 being the total numbers of the particles in BECs, α_0 being the dimensionless spatial unit, and $\alpha_{kl}=\alpha_{lk}$ being the s-wave scattering lengths between the l-*th* and k-*th* components (positive for repulsive interaction and negative for attractive interaction). L_z is the rotating angular momentum with rotating speed Ω, defined as

$$L_z=i\Omega(y\partial_x-x\partial_y).$$

λ is the effective Rabi frequency to realize the internal atomic Josephson junction by a Raman transition. If there is no effective Rabi frequency and no rotation, i.e., $\lambda=0$ and $\Omega=0$, the CGP equations collapse to the coupled nonlinear Schrödinger equations, they have been discussed by many authors [21].

Under appropriate assumption, to make the problem well-posed, with the following initial data

$$u(x,y,0)=u_0(x,y),\quad v(x,y,0)=v_0(x,y),\tag{1.2}$$

the problem (1.1a)-(1.2) satisfies the following laws [5].

- The component-masses

$$\mathcal{M}_u(t)=\int_{\mathbb{R}^2}|u(x,y,t)|^2dxdy\quad\text{and}\quad\mathcal{M}_v(t)=\int_{\mathbb{R}^2}|v(x,y,t)|^2dxdy$$

with respect to the u-component and v-component particles, respectively, satisfy the equalities

$$\frac{d}{dt}\mathcal{M}_u(t)=-\lambda\int_{\mathbb{R}^2}\left(v(x,y,t)\overline{u(x,y,t)}-u(x,y,t)\overline{v(x,y,t)}\right)dxdy,\tag{1.3a}$$

$$\frac{d}{dt}\mathcal{M}_v(t)=-\lambda\int_{\mathbb{R}^2}\left(u(x,y,t)\overline{v(x,y,t)}-v(x,y,t)\overline{u(x,y,t)}\right)dxdy.\tag{1.3b}$$

Therefore, the total mass of the particles of the system is an invariant, that is,

$$\mathcal{M}(t)=\mathcal{M}_u(t)+\mathcal{M}_v(t)=\mathcal{M}(0).\tag{1.4}$$

Furthermore, $\mathcal{M}_u(t)=\mathcal{M}_u(0)$ and $\mathcal{M}_v(t)=\mathcal{M}_v(0)$ in case of $\lambda=0$. This indicates that the particle masses of each component are conserved.