Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 6, pp. 1327-1352

DOI: 10.4208/aamm.OA-2018-0266
December 2020

Complex Pattern Formations by Spatial Varying Parameters

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Received 25 December 2018; Accepted (in revised version) 6 December 2019

Abstract. Pattern formations by Gierer-Meinhardt (GM) activator-inhibitor model are considered in this paper. By linear analysis, critical value of bifurcation parameter can be evaluated to ensure Turing instability. Numerical simulations are tested by using second order semi-implicit backward difference methods for time discretization and the meshless Kansa method for spatially discretization. We numerically show the convergence of our algorithm. Pattern transitions in irregular domains are shown. We also provide various parameter settings on some irregular domains for different patterns appeared in nature. To further simulate patterns in reality, we construct different kinds of animal type domains and obtain desired patterns by applying proposed parameter settings.

AMS subject classifications: 35K57, 65M70

Key words: Gierer-Meinhardt model, pattern formation, meshless method, spatially varying parameter.

1 Introduction

Many biological process, including animal pigmentation [8, 33, 35, 36], tumor formation [16] and animal population distribution [9], can be modeled by nonlinear reaction diffusion systems. Mathematical analysis and simulation can help researchers better understanding interaction between chemicals in animal growth from a single cell to large amount patterns we see in nature. In 1952, Turing proposed the first model for morphogens phenomenon in his paper [40]. In the paper, he studied mathematical models, which can generate the organism structure from a zygote. Prompted by this study, many Turing models were put forward for different biological and physical process in nature.

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In 1955, a mathematical model of some threshold phenomenon in the nerve membrane was posed by Fitzhugh in [19]. In 1979, Schnakenberg [39] proposed a model to describe trimolecular autocatalytic reactions with two chemicals. The Gierer-Meinhardt activator-inhibitor model was first introduced by Gierer and Meinhardt in [20] to describe the spatial pattern of tissue structures starting from almost homogeneous tissue.

Most Turing models can be written by reaction diffusion systems with two chemicals as

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \Delta u + f(u,v), \\ \dfrac{\partial v}{\partial t} = D_v \Delta v + g(u,v), \end{cases} \quad \text{in } \Omega, \tag{1.1}$$

with $u(x,t)$ and $v(x,t)$ denoting concentrations of two chemicals at spatial position x and time t, D_u and D_v being diffusion constants. Functions $f(u,v)$ and $g(u,v)$ have different representations in different models and they describe the reactions of two chemicals. In this paper, we consider the Gierer-Meinhardt (GM) model [30] with

$$\begin{cases} f(u,v) = k_1 + k_3 \dfrac{u^2}{v} - k_2 u, \\ g(u,v) = k_4 u^2 - k_5 v, \end{cases}$$

for some positive parameters $k_1, \cdots, k_5$. By the manipulation in [34], the GM model can be written in a non-dimensional form

$$\begin{cases} f(u,v) = r\left(1 + \dfrac{u^2}{v}\right) - \mu u, \\ g(u,v) = r u^2 - \nu v. \end{cases} \tag{1.2}$$

where r, ν and μ are positive constants.

Based on some linear analysis, we will study Turing instability of the Gierer-Meinhardt model for fixed bifurcation parameter. Numerical study for pattern formation is also essential since the non-uniform solution corresponding to spatial patterns cannot be found analytically. For time discretization, numerical schemes can be chosen with different convergence behavior. The Runge-Kutta method used in [13, 38] and the Crank-Nicolson scheme was employed in [22, 23]. The alternating direction implicit Crank-Nicholson (ADI-CN) was applied in [17] to solve two dimensional Riesz space fractional diffusion equations. In [18], the authors proposed a two level method for semi-linear reaction-diffusion equations. We use the implicit-explicit SBDF2 scheme [37] in this paper which also be employed in [11]. Different numerical methods are also proposed for the spatial discretization, such as the finite difference method [8, 35, 36], the finite element method [28, 41] and different kinds of meshless methods (the element free Galerkin method [14, 29], the local radial basis function method [38]), ect. Compared to methods with mesh, meshless methods are easier applied to irregular domains, for instance ellipse domains and butterfly shape domains considered in [29]. Kansa method, proposed by E.