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Eigenvalues of a Differential Operator and Zeros of the Riemann ζ-Function

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. The eigenvalues of a differential operator on a Hilbert-Pólya space are determined. It is shown that these eigenvalues are exactly the nontrivial zeros of the Riemann ζ-function. Moreover, their corresponding multiplicities are the same.

Key Words: Hilbert-Pólya space, zeros of zeta function, differential operator, eigenvalue.

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1 Introduction and preliminaries

In studying the Riemann hypothesis it is important to establish a connection between spectrum of operators and zeros of the Riemann zeta function. This is our continued effort for finding such an interplay.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$. Functions in $\mathcal{S}(\mathbb{R})$ are smooth and rapid decreasing when $|x| \to \infty$. That is,

$$\mathcal{S}(\mathbb{R}) = \Big\{ f \in C^\infty(\mathbb{R}) \Big| \sup_{x\in\mathbb{R}} |x^m f^{(n)}(x)| < \infty \text{ for } m, n \in \mathbb{N} \Big\},$$

where $\mathbb{N} = \{0, 1, \cdots\}$. Denote by $\mathcal{H}_\cap$ the subspace of all even functions f in $\mathcal{S}(\mathbb{R})$ with $f(0) = \mathfrak{F}f(0) = 0$. Here

$$\mathfrak{F}f(x) = \int_{-\infty}^{\infty} f(y) e^{-2\pi i x y} dy$$

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is the Fourier transform of f. In this paper, we may identify functions in $\mathcal{H}_\cap$ with their restrictions on $(0,\infty)$.

Denote by $\mathcal{H}_-$ the set of all smooth functions on $(0,\infty)$ that decrease rapidly when $x\to 0+$ and ∞. Specifically,

$$\mathcal{H}_- = \Big\{ f\in C^\infty(0,\infty) \Big| \sup_{x>0} |x^m f^{(n)}(x)| < \infty \text{ for } |m|, n\in\mathbb{N} \Big\}.$$

For $f\in\mathcal{H}_\cap$ we denote by Z the action given by

$$Zf(x)=\sum_{n=1}^{\infty} f(nx), \quad x>0.$$

By Poisson's summation formula, $Z\mathcal{H}_\cap \subseteq \mathcal{H}_-$. The quotient space $\mathcal{H}_-/Z\mathcal{H}_\cap$, denoted by $\mathcal{H}$, is regarded as a Hilbert-Pólya space, although such a quotient does not carry a Hilbert space structure.

The fundamental differential operator D on $\mathcal{H}_-$ is given by

$$Df(x)=-xf'(x), \quad f\in\mathcal{H}_-.$$

Notice that $D\mathcal{H}_- \subseteq \mathcal{H}_-$ and $D(Z\mathcal{H}_\cap)\subseteq Z\mathcal{H}_\cap$. The operator D thus induces a differential operator on $\mathcal{H}$, which we denote by D_-. We follow the notation introduced in [5]. For basics on the Riemann zeta function $\zeta(s)$, we refer to [6].

The Hilbert-Pólya conjecture states that the non-trivial zeros of the Riemann ζ-function correspond (in certain canonical way) to the eigenvalues of some operator, and Riemann Hypothesis is equivalent to the self-adjointness of the operator.

In the direction of the Hilbert-Pólya conjecture, a spectral interpretation for zeros on the critical line $\mathrm{Re}(s)=\frac{1}{2}$ was given by A. Connes [1]. He constructed a closed, densely defined, unbounded differential operator D_χ and a Hilbert-Pólya space $\mathcal{H}_\chi$. His operator D_χ has discrete spectrum, which is the set of imaginary parts of critical zeros of the L-function with Grössencharacter χ [1, Theorem 1, pp. 40].

In [2], L. Ge studied the action of the differential operator D on a Hilbert-Pólya space, which is slightly different from $\mathcal{H}$, and obtained that the point spectrum of D coincides with the nontrivial zeros of $\zeta(s)$.

Meyer proved in [5, Corrollary 4.2, pp. 8] that the eigenvalues of the transpose D_-^t of D_- (acting on the space of continuous linear functionals on $\mathcal{H}$) are exactly the non-trivial zeors of $\zeta(s)$, and the algebraic multiplicity of an eigenvalue is equal to the vanishing order of the corresponding zero of $\zeta(s)$.

Li [4] proved by an explicit construction that the nontrivial zeros of $\zeta(s)$ are the eigenvalues of D_- acting on $\mathcal{H}$ (instead of D_-^t in [5]), which we state as follows (Theorem 1.1 and Theorem 1.4 in [4]).

Theorem 1.1 ([4]). *Let ρ be a nontrivial zero of $\zeta(s)$. Let F_ρ be the function given by*

$$F_\rho(x)=\int_1^\infty Z\eta(tx)t^{\rho-1}dt,$$