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Perturbations of Multipliers of Systems of Periodic Ordinary Differential Equations

Leonid Berezansky*, Michael Gil' and Liora Troib

Department of Mathematics, Ben Gurion University of the Negev P.O. Box 653, Beer-Sheva 84105, Israel

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Abstract. The paper deals with periodic systems of ordinary differential equations (ODEs). A new approach to the investigation of variations of multipliers under perturbations is suggested. It enables us to establish explicit conditions for the stability and instability of perturbed systems.

AMS subject classifications: 34D10, 34D20

Key words: Periodic systems, perturbations, stability, instability.

1 Introduction

This paper deals with perturbations of multipliers and stability of vector linear ODEs with periodic matrix coefficients. The problem of stability analysis of periodic systems continues to attract the attention of many specialists despite its long history. It is still one of the most burning problems of the theory of ODEs, because of the absence of its complete solution. The classical results on periodic systems are presented in the well-known books [2, 5, 9]. The recent investigations of stability of linear and nonlinear periodic systems and periodic solutions can be found in the very interesting papers mentioned below.

Zevin considers in [11] a periodic canonical system. He proposes a new definition of the index of stability domains of the system and presents a simple proof for the Helfide-Lidskij theorem on the structure of stability domains. The directed convexity of stability domains is also discussed. In the paper [10], Zevin constructs a stability theory for canonical systems in terms of the index function. His approach allows us to solve a series of problems from the periodic system theory, in particular, the problem of strong stability condition; estimation of stability domains of parametric oscillations;

*Corresponding author.
URL: `http://www.math.bgu.ac.il/ brznsky/index.html`
Email: brznsky@cs.bgu.ac.il (L. Berezansky), gilmi@cs.bgu.ac.il (M. Gil'), liora@cs.bgu.ac.il (L. Troib)

parametric stabilization of unstable systems. The problem on parametric stabilization of the upper equilibrium of a pendulum is considered as an example. The paper [6] should be mentioned. It deals with perturbations of some nonautonomous oscillatory canonical systems with a small parameter. The continuous dependence of periodic solutions for the periodic quasilinear ordinary differential system containing a parameter is established in the paper [7]. The authors of the paper [1] review results on the exponential stability of nonautonomous linear periodic evolution equations. In the paper [8], Chebyshev polynomials are utilized to investigate the solutions higher order scalar linear differential equations with periodic coefficients. As it is well-known, Lyapunov has obtained conditions on the real periodic function $q(t)$ under which the second-order differential equation

$$y'' + q(t)y = 0,$$

is stable. In the paper [4] the authors generalize Lyapunov's result for the differential equation of the form

$$(p(t)y')' + q(t)y = 0,$$

with periodic coefficients $p(t)$ and $q(t)$. Certainly we could not survey the whole subject here and refer the reader to the above listed publications and references given therein.

Furthermore, as it is well-known [9, pp. 282], the classical methods of the perturbation theory of periodic systems is based on the expansions of the perturbed evolution operator in fractional powers of the perturbation parameter. Such methods often require cumbersome calculations. We suggest a new approach to the investigations of perturbations of multipliers which is based on the recent estimates for the norm of the resolvent of a matrix. Our results enable us to establish explicit conditions for stability and instability of perturbed systems. The Hill equations are considered as examples.

2 The basic lemma

Consider the equations

$$\dot{x} = A(t)x, \tag{2.1}$$

$$\dot{x} = \widetilde{A}(t)x, \tag{2.2}$$

where $A(t)$ and $\widetilde{A}(t)$ are T-periodic piecewise continuous $n \times n$-matrices.

Let $U(t)$ and $\widetilde{U}(t)$ be the Cauchy operators to Eqs. (2.1) and (2.2), respectively. Then

$$U(t,s) = U(t)U^{-1}(s) \quad \text{and} \quad \widetilde{U}(t,s) = \widetilde{U}(t)\widetilde{U}^{-1}(s),$$

are the corresponding evolution operators. The eigenvalues μ and $\widetilde{\mu}$ of $U(T)$ and of $\widetilde{U}(T)$, respectively, taken with their multiplicities are called the multipliers to Eqs. (2.1) and (2.2), respectively. Denote

$$\gamma := \|U(T) - \widetilde{U}(T)\|,$$