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Evolutionary Systems Representations Based on Later Time Samples and Applications to PDEs

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Abstract. Let $\{f_n\}_{n=1}^{\infty}$ be a basis for $L_2([0,1])$ and $\{g_n\}_{n=1}^{\infty}$ be a system of functions of controlled decay on $[0,\infty)$. Considering a function $u(x,t)$ that can be the represented in the form

$$u(x,t)=\sum_{n=1}^{\infty}a_nf_n(x)g_n(t),$$

where $a_n\in\mathbb{R}$, $x\in[0,1]$ and $t\in[0,\infty)$, we investigate whether the function $f(x)=u(x,0)$ can be approximated, in a reasonable sense, by using data $u(x_0,t_1),u(x_0,t_2),\ldots,$ $u(x_0,t_N)$. A mathematical framework and efficient computational schemes are developed to determine approximate solutions for various classes of partial differential equations via sampled data by first establishing a near-best approximation of f.

AMS subject classifications: 35R30, 35A25

Key words: Dynamical system, evolutionary systems representation, near-best approximation, initial data, PDEs.

1. Introduction

Efficient data processing is essential in large data applications, whether the phenomenon of interest is sound, heat, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. The spatial/time distribution of these aspects can be described similarly in terms of PDEs. When solving a PDE of interest, we need to know the initial conditions, described by some functions. However, in real-life applications, full knowledge of the initial conditions is often impossible due to unavailability of a large number of sensors. The way to overcome this problem is to exploit the evolutionary nature of the sampling environment for a reduced number of sensors — i.e. to employ the concept of dynamical sampling. Dynamical sampling problems have become an important issue in many areas in theory and

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applications of mathematics and engineering. The problem of estimating sources from the spatiotemporal samples taken by a network of spatially distributed sensors has been considered by Vetterli and Lu [10] and their work has motivated a branch of dynamical sampling research [3–5]. A typical dynamical sampling problem is to find special locations that allow one to recover an unknown function from the samples at these locations. Dynamical sampling uses samples from various time levels for a single reconstruction, which departs from classical sampling theory where a signal does not evolve in time [3]. Recently, Aldroubi and Petrosyan have studied dynamical sampling using bases and other systems [3–5]. Additionally, the classical problem of inverse heat conduction [6, 7] has been revisited by DeVore and Zuazua [9] who investigated how to recover the initial temperature distribution of a finite body from temperature measurements made at a fixed number of later times. Besides Aceska *et al.* considered the number of measurements needed to recover the initial profile with a prescribed accuracy and provided an optimal reconstruction algorithm under the assumption that the initial profile is in a Sobolev class. Inspired by the works [1, 8, 9], in this paper we study conditions on an evolving system and spatial samples in a more general set up using bases. Specifically, we study whether the series representation

$$u(x,t)=\sum_{n=1}^{\infty} a_n f_n(x) g_n(t)$$

can be reasonably approximated through dynamical samples. The concept of dynamical sampling is beneficial in setups where the available sensing devices are limited due to access constraints. In such an under-sampled case, we use the coarse system of sensors multiple times to compensate for the lack of samples at a single time instance. Our focus is on developing methods which efficiently approximate solutions of important PDEs by engaging the dynamical nature of the setup dictated by the initial conditions. We develop a theory and algorithms for a new sampling and approximation framework. This framework combines spatial samples of various states of approximations and eventually reconstructs the solution exactly. We assume that the initial state of the solution is in a selected Sobolev class.

2. Approximation Scheme

We study the problem of solving an initial value problem (IVP) from discrete measurements made at later times; thus, the initial conditions are not known in full details. We aim to show that with a carefully selected placement of the sensing devices, the unknown initial conditions can be completely determined by later time measurements. Thus a general solution to the IVP of interest is derived. We study a broad range of PDEs, describing the IVP, and focus on initial conditions functions (ICF), with suitable decompositions, under the assumption that the ICF is in a select function class. Further on, we study the number of measurements that are needed to recover the solution to a desired accuracy, and we conduct research to optimize the reconstruction algorithm.

Let $\{f_n\}_{n=1}^{\infty}$ be a basis for $L_2([0,1])$. We consider an unknown function $u(x,t)$ which