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# Review Article

# A Unified Approach to Solving Some Inverse Problems for Evolution Equations by Using Observability Inequalities

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**Abstract.** We survey some of our recent results on inverse problems for evolution equations. The goal is to provide a unified approach to solve various types of evolution equations. The inverse problems we consider consist in determining unknown coefficients from boundary measurements by varying initial conditions. Based on observability inequalities and a special choice of initial conditions, we provide uniqueness and stability estimates for the recovery of volume and boundary lower order coefficients in wave and heat equations. Some of the results presented here are slightly improved from their original versions.

**AMS subject classifications**: 35R30

**Key words**: Evolution equations, Laplace-Beltrami operator, observability inequality, geometric control, initial-to-boundary operator.

## 1 Introduction

Inverse coefficient problems for evolution equations have been a very active area in mathematical and numerical research over the last decades, driven by numerous applications. They are intrinsically difficult to solve: this fact is due in part to their very mathematical structure and to the fact that generally only partial data is available [17]. We survey in

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this paper some of our recent results on inverse problems for evolution equations concerning heat and wave equations. In [2] the authors proposed a general method to deal with inverse source problems for evolution equations. Starting from the ideas in [2], we developed an approach based on observability inequalities and a spectral decomposition to solve some inverse coefficients problems in evolution equations [3–5]. However the approach is older than that. Inverse coefficient problems in heat and wave equations using control techniques have been studied by a large community of people (see for instance [6, 8, 15, 16, 21, 24, 25, 29] and the references therein). It would be impossible to present here all the relevant results that have been proved in this research direction. We will be mainly focusing on the results that are closely connected to the considered inverse coefficient problems in heat and wave equations.

The measurements are made on a sub-boundary by varying initial conditions. The key idea in our analysis consists in reducing the inverse coefficients problems to inverse source problems. This is achieved by using a spectral decomposition and unique continuation property of eigenfunctions.

For simplicity convenience we limited ourselves to initial boundary value problems for wave and heat equations. But our analysis can be extended to other types of evolution equations such as dynamical Schrödinger equation.

The main ingredient in our approach is observability inequalities. We point out that the wave and the heat equations have different observability properties. We know that, under some appropriate conditions, the wave equation is exactly observable, while the heat equation is only final time observable [28, 30]. We refer to Section 2 for details. In Section 3, we establish weighted interpolation inequalities involving the eigenfunctions of Laplace-Beltrami operator that are useful in the analysis of the stability issue of the studied inverse coefficient problems. These inequalities have been obtained by quantifying the unique continuation property for the Laplace-Beltrami operator through weighted energy estimates with the aid of Carleman type inequalities. We present an abstract framework for the inverse source problem in Section 4. Based on the introduced observability inequalities we provide uniqueness and stability inequalities of the recovery of volume and boundary lower order coefficients in wave and heat equations from boundary measurements in respectively Sections 5 and 6.

## 2 Observability inequalities

We collect in this section various observability inequalities that are necessary to the analysis of the inverse problems we want to tackle in this text. Since most of these results are well recorded in the literature we limited ourselves to give their precise statement and provide the references where the proofs can be found.