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Extrapolation Accelerated PRESB Method for Solving a Class of Block Two-By-Two Linear Systems

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Abstract. We use extrapolation acceleration technique to speed up the preconditioned square block matrix splitting iteration method for two-by-two block linear systems. It is shown that for relaxation parameter $\omega = 4/3$, the convergence factor of the iteration method under consideration is $1/3$. This yields the robustness and efficiency of the method. Numerical examples confirm the theoretical results and demonstrate the effectiveness of the approach developed.

AMS subject classifications: 65F10, 65F50, 65W05

Key words: PRESB preconditioner, two-by-two block matrix, spectral radius, iteration method, convergence factor.

1. Introduction

We consider the block two-by-two linear systems

$$\mathscr{A}\tilde{z} = \begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1.1}$$

where $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices such that

$$\mathrm{null}(W) \cap \mathrm{null}(T) = \{0\}. \tag{1.2}$$

Various applications involving systems (1.1) include time-dependent PDEs, distributed control problems, structural dynamics, quantum mechanics, molecular scattering and the solution of complex-valued systems

$$(W + \mathrm{i}T)(x + \mathrm{i}y) = f + \mathrm{i}g$$

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in real arithmetics — cf. Refs. [1–6, 8, 13, 19, 20, 22, 23, 26, 30, 34]. Therefore, such linear systems attracted considerable attention in recent years. A variety of efficient iteration methods and preconditioning techniques for solving systems (1.1) have been developed [2, 3, 5–13, 17, 18, 24, 25, 27, 28, 31–33, 35, 37–41]. Among those, preconditioned MHSS (PMHSS) iterations [12], preconditioned GSOR (PGSOR) iterations [24] and a preconditioned square block (PRESB) method [5, 28] proved to be very efficient and robust. In particular, Bai *et al.* [14] used Hermitian and skew-Hermitian splitting of the coefficient matrix $\mathscr{A}$ to prove that the spectral radius of the PMHSS iteration matrix is bounded by $\sigma(\alpha) = \sqrt{\alpha^2+1}/(\alpha+1)$ and $\sigma(\alpha)$ attains the minimum $\sqrt{2}/2 \approx 0.707$ at $\alpha = 1$. However, it was noted that the PMHSS iteration matrix has complex eigenvalues. Hezari *et al.* [24] estimated the convergence factor of the PGSOR method by $(\sqrt{2}-1)/(\sqrt{2}+1)$. The PGSOR method involves computing optimal parameters α_{opt} and ω^* related to the smallest and largest eigenvalues of $W^{-1}T$. This method can be regarded as a special case of the GSOR method from [16]. In addition, Axelsson [5] proposed a PRESB preconditioner — viz

$$P(\alpha) = \begin{pmatrix} W & -T \\ T & \alpha^2 W + 2\alpha T \end{pmatrix}.$$

It is shown that $P(\alpha)^{-1}\mathscr{A}$ has only real valued eigenvalues, an optimal parameter α^* is obtained and the eigenvalues of the PRESB preconditioned matrix are bounded by $[1/2, 1]$ with $\alpha = 1$. However, this optimal parameter α^* is related to restrictive eigenvalues of the matrix $W^{-1}T$, that may be not available for large scale problems.

It is well known that extrapolation can be used to improve the convergence of stationary iteration methods [15, 37]. Nevertheless, it is difficult to apply since a relaxation parameter should be determined and the spectral properties of the iteration matrix can be analysed under stronger conditions. In other words, after the extrapolation the iteration method may require a more complex theoretical analysis. For example, in order to find a quasi-optimal relaxation parameter for the SSOR iteration method, the coefficient matrix should be symmetric and positive definite. Theretofore, not all stationary methods can be extrapolated.

Let us note that the extrapolation accelerated technique does not work for PMHSS and PGSOR methods since the corresponding iteration matrices have complex eigenvalues. On the other hand, the eigenvalues of the PRESB iteration matrix are real valued and are restricted to the interval $[0, 1/2]$ with $\alpha = 1$. This motivates us to utilise the extrapolation accelerated technique to speed up the PRESB method. The new method, called EAPRESB, is faster than PRESB and PMHSS methods. We also can use the quasi-optimal parameter $\omega = 4/3$ to make the convergence factor $1/3$ to be exploited in all examples. Another advantage of the EAPRESB method is that the matrices W and T should not be positive definite simultaneously.

The rest of this paper is organised as follows. In Section 2, we review the PRESB method, introduce the extrapolation accelerated PRESB iteration method and discuss its convergence. Section 3 contains the results of numerical experiments. Finally, our brief conclusions are given in Section 4.