Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 4, pp. 1008-1034

DOI: 10.4208/aamm.OA-2019-0081
August 2020

Stable Mixed Element Schemes for Plate Models on Multiply-Connected Domains

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Received 28 March 2019; Accepted (in revised version) 16 September 2019

Abstract. In this paper, we study the mixed element schemes of the Reissner–Mindlin plate model and the Kirchhoff plate model in multiply-connected domains. Constructing a regular decomposition of $H_0(\text{rot},\Omega)$ and a Helmholtz decomposition of its dual, we develop mixed formulations of the models which are equivalent to the primal ones respectively and which are uniformly stable. We then present frameworks of designing uniformly stable mixed finite element schemes and of generating primal finite element schemes from the mixed ones. Specific finite elements are given under the frameworks as an example, and the primal scheme obtained coincides with a Durán-Liberman scheme which was constructed originally on simply-connected domains. Optimal solvers are constructed for the schemes.

AMS subject classifications: 65N30, 74K20

Key words: Reissner–Mindlin plate, Kirchhoff plate, multiply-connected domain, uniformly optimal solver, regular decomposition.

1 Introduction

In this paper, we study the Reissner–Mindlin model for moderately thick plates and the Kirchhoff model for thin plates on multiply-connected domains. Among the various kinds of plate models in structural analysis, these two fall in the most frequently used ones for thick and thin plates, respectively. It is known that the stability of the Reissner–Mindlin model in its primal formulation is of a complicated representation, and we will discuss the mixed element scheme of the Reissner–Mindlin model. Meanwhile, the Kirchhoff model falls into the category of fourth order elliptic problem, and there have been many primal finite element methods for that. However, mixed element discretisation can bring in flexibility in implementation by finite element package and designing multilevel

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methods. In this paper, we will also present mixed element schemes for the Kirchhoff model.

There have been large literature on the mathematical analysis and numerical methods on Reissner–Mindlin model; we refer to [20] for a brief review. The mathematical analysis and numerical solution of the model constructed on convex simply-connected polygons have been studied well. Procedures for developing stable and convergent mixed and then primal finite element methods have been firmly established since [3,15–17]. The fast solution of the generated mixed and primal finite element system is discussed in [5]. For these achievements, mixed formulations played fundamental roles [3,16,17]. Some other mixed formulations can be found in, e.g., [1,9,22,28,39].

In contrast, when the domain is multiply-connected (thus non convex), to the author's knowledge, the mixed formulations of the models have hardly been discussed though it is useful in practice. An important difference between multiply- and simply- connected domains is that a space of harmonic functions is contained in the kernel subspace of rot in the space $H_0(\mathrm{rot})$, which affects the regular decomposition of $H_0(\mathrm{rot},\Omega)$ and the Helmholtz decomposition of $(H_0(\mathrm{rot},\Omega))'$, and procedures developed in [3, 15–17] whereas the stable decompositions play a crucial role can not immediately be repeated. Indeed, because of the harmonic functions, it remains open how to figure out the space $\nabla H^1_0(\Omega)+(H^1_0(\Omega))^2$ which has shown to be the fundamental and the starting point of designing stable mixed formulation of the Reissner-Mindlin and thus Kirchhoff plate models. The influence of the existence of harmonic functions has been discussed in the context of Maxwell equation, for which we refer to, e.g., [13,32] for related discussion, while its influence on Reissner–Mindlin plate has not been discussed. Some investigation on the problem will be carried on in the present paper.

In this paper, it is firstly verified that $H_0(\mathrm{rot},\Omega)=\nabla H^1_0(\Omega)+(H^1_0(\Omega))^2$ still holds on multiply-connected domains, namely the harmonic functions do not have to appear in the decomposition, though they are generally expected to. Based on this basic fact, we present a new mixed formulation for the Reissner–Mindlin model, and prove its uniform stability. This is basically the main new result of the present paper. Then, a framework of constructing mixed finite element schemes is presented. The error estimation in energy norm follows with respect to the regularity of the system. Based on the uniform stability, an optimal diagonal preconditioner is designed which consists of solving Poisson systems and $H(\mathrm{rot})$ systems. By Hiptmair-Xu decomposition and the techniques of auxiliary space preconditioning [25,27,40,42], the implementation of the preconditioner is finally transferred to solving several discretized Poisson equations, which can be carried out by existing optimal solvers. This way, the mixed Reissner-Mindlin discretization can be solved optimally fast by running a preconditioned MinRes method [37]. So far as we know, there has been no discussion on the optimally fast solver for the Reissner-Mindlin plate model on multiply-connected domains. Particularly, all direct multigrid methods on the primal discretizations known to us will assume full regularity of the solutions [12, 33, 38], which limits its utilization on even nonconvex simply-connected domains. A specific example of the finite elements is given under the framework.