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A Fourth Order WENO Scheme for Hyperbolic Conservation Laws

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Abstract. In this work, a fourth order weighted essentially non-oscillatory (WENO) scheme is developed for hyperbolic conservation laws. The new reconstruction is a convex combination of three linear reconstructions. To keep high order accuracy in smooth regions and maintain non-oscillatory near discontinuities, the nonlinear weights are carefully designed. The main advantage of the proposed scheme is that the scheme achieves one order of improvement in accuracy in smooth regions compared with the classical third order scheme when using the same spatial nodes. Several benchmark examples are presented to verify the scheme's fourth order accuracy and capacity of dealing with problems containing complicated structures.

AMS subject classifications: 65M06, 35L65

Key words: Hyperbolic conservation laws, WENO scheme, nonlinear weights, Euler equations.

1 Introduction

Hyperbolic conservation laws are one of the classical topics of interest in the present days. Since they have extensive applications in scientific and engineering areas, tremendous efforts have been devoted to designing numerical methods for them. Solutions to hyperbolic conservation laws can contain rich structures such as strong shocks, contact discontinuities and various smooth structures. A successful numerical method should capture shocks with high resolution, avoid spurious oscillations near discontinuities and achieve high order accuracy in smooth regions. There are several different types of methods: total variation diminishing schemes (TVD), weighted essentially non-oscillatory schemes (WENO) and discontinuous Galerkin schemes (DG) [14–18], etc. Among the variety of numerical methods, WENO schemes have played an indispensable role and enjoyed widespread popularity.

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In 1994, Liu et al. proposed the first version of WENO schemes on the uniform meshes [10]. The key idea of WENO scheme is a linear combination of lower order reconstructions to obtain a higher order approximation. Based on the nonlinear weights mechanism, WENO scheme would automatically achieve high order accuracy in smooth regions of the solution and guarantee a non-oscillating property near discontinuities. After the pioneer work of Liu et al., WENO scheme got into a period of vigorous advancement. Jiang and Shu improved the original WENO scheme and established a general framework to design smoothness indicators and nonlinear weights [8]. Since the scheme of Jiang and Shu may lose accuracy at critical points, Henrick et al. developed a mapping technique to overcome this problem [5]. Unfortunately, the mapping procedure is computationally expensive. Borges et al. introduced a global high order reference value and modified the weighting formulation [2]. The resulting WENO-Z scheme is efficient and can achieve fifth order accurate at critical points. Very high order WENO schemes were presented in the literature [4]. It should be pointed out that negative weights may appear when construct a high order WENO scheme with an irregular partition. To deal with this, one can employ a splitting technique [12]. Besides, Zhu and Qiu developed a new simple WENO scheme and Balsara et al. presented an efficient class of WENO schemes with adaptive order recently [1,23]. The main merit of their schemes is that the linear weights can be artificially set to be any random positive numbers with the only requirement that their sum equals one. We do not mention all WENO schemes here. One can consult [3,9,13,15] and the reference therein for further understanding.

Most of the existing rth order finite difference WENO schemes compute $f^{\pm}_{i+1/2}$ on two different r-points stencils. In other words, they use $r+1$ points to obtain $f_{i+1/2}$. From the knowledge of numerical analysis, we know that using $r+1$ points can achieve a $(r+1)$th order approximation. In view of this, some researchers have attempted to improve the accuracy of WENO scheme. By using the same spatial nodes as the classical fifth order WENO scheme in [8], Hu et al. developed an adaptive central-upwind sixth order WENO scheme in which the linear weights are unique [6]. Recently, Huang and Chen developed a new adaptive central-upwind sixth order WENO scheme in which the linear weights can be any positive numbers with the requirement that their sum equals one [7]. Another version of the improvement was presented in [22] by utilizing a symmetrical stencil instead of two upwind stencils.

In this paper, we also concentrate on the improvement of WENO scheme by presenting a fourth order WENO scheme. First, we construct an optimal polynomial of degree three based on the big central stencil and three linear polynomials on corresponding substencils. Then the approximation of $f^{\pm}_{i+1/2}$ obtained from the optimal polynomial can be written as convex combinations of the values of three polynomials at the points $x_{i+1/2}$ with linear weights. To suppress the spurious oscillations and improve the resolution in discontinuous regions, the linear weights are modified to the nonlinear weights based on the new reference smoothness indicator. Comparing with the classical third order WENO scheme of Jiang and Shu [8], this new scheme adds a linear approximation of $f^{\pm}_{i+1/2}$ in the reconstruction procedure. Actually, this linear approximation is only active to recover