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New Mixed Finite Volume Spaces for Elliptic Problems on Parallelepiped

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Abstract. In this paper, we define a new nonconforming finite element space on parallelepiped. Using our new nonconforming space and a vector part of Kim-Kwak mixed finite element space, we suggest a new class of higher order mixed finite volume method. We show that the mixed finite volume methods can be implemented by solving the primal problem with our new nonconforming finite element methods for the pressure variable. And we can obtain the velocity variable by local recovery technique. An optimal error analysis is given and also numerical results are presented to support our analysis.

AMS subject classifications: 65N15, 65N30, 35J60

Key words: Mixed finite volume methods, nonconforming finite element methods, mixed finite element methods, error analysis.

1 Introduction

The mathematical analysis and applications of mixed finite element methods (MFEMs) have been widely developed since the seventies [1–5, 15, 16]. The main idea of the mixed methods is to use the Darcy velocity as a new variable and change the given equation into a system of differential equations. By discretizing this system, we can calculate two variables, the velocity and the pressure, simultaneously. It is well known that in many cases mixed finite element methods give better approximations for the velocity variable than classical Galerkin methods [2, 5, 16]. However, the mixed formulation is more difficult to handle because the resulting system has many more variables and causes a saddle point problem. Also, it is more expensive from a computational point of view. Thus there are restrictions on the use in industry.

For the lowest order case, it is well known that there exist equivalent nonconforming finite element spaces associated with mixed finite element spaces [2, 6, 7]. However, for

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higher order cases, the corresponding nonconforming methods involve projections into the vector part of the mixed finite element spaces, which are not easy to deal with. There have been some efforts to overcome this difficulty. For example, Croisille et al. [9–11] and Kwak [8, 14] used a mixed finite volume type of methods (MFVMs). The main idea is to integrate the system over each element with certain test functions. In particular, Kwak suggested a new class of mixed finite volume methods under a series of hypotheses and showed that they are equivalent to some nonconforming finite element methods (NFEMs) [14]. And the velocity variable are obtained cheaply using recovery technique. For the three dimensional problem, they used the velocity variable lying in the Nedelec spaces [15] and the pressure variable lying in the nonconforming finite element spaces introduced by Arbogast and Chen [1].

In this paper, we define a new nonconforming finite element space on parallelepiped. Using this space, we introduce a new class of mixed finite volume methods of higher order with fewer degrees of freedom than the existing one. To construct such elements, we eliminate some redundant elements from existing nonconforming element spaces of [1] and provide proper degrees of freedom to ensure unisolvence. For the velocity variable, we use Kim-Kwak element spaces [13] which have smaller number of degrees of freedom than the Nedelec spaces. Thus we can recover velocity variables from the solution of NFEMs with fewer variables. We prove the optimal error estimates of our new scheme and present the numerical results to confirm the theory.

The outline of this paper is as follows: in Section 2, we present the model problem used in this paper and explain that the MFVMs for elliptic problems can be easily implemented by solving the primal problem with some nonconforming finite element methods. In Section 3, we define a new nonconforming finite element spaces that match the mixed finite element space in [13]. An error analysis are carried out in Section 4 and numerical results are shown in Section 5. Finally, a short conclusion is given in Section 6.

2 Governing equations and the mixed finite volume methods

We consider the following second order elliptic boundary value problem:

$$\begin{cases} -\mathrm{div}(\kappa\nabla p)+cp=f & \text{in } \Omega, \\ p=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

over a bounded Lipschitz domain Ω in $\mathbb{R}^3$. Where κ is a symmetric, positive definite matrix and $c(x)\geq 0$, $c\in L^\infty(\Omega)$. We will assume that f is a given function in $H^k(\Omega)$ for some integer $k\geq 0$ to explain the higher order methods. Let us introduce a vector variable $\mathbf{u}=-\kappa\nabla p$. Then the model problem (2.1) can be rewrited to give the following first order system :

$$\begin{cases} \mathbf{u}+\kappa\nabla p=0 & \text{in } \Omega, \\ \mathrm{div}\mathbf{u}+cp=f & \text{in } \Omega, \\ p=0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$