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Boundedness of High Order Commutators of Riesz Transforms Associated with Schrödinger Type Operators

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Abstract. Let $\mathcal{L}_2 = (-\Delta)^2 + V^2$ be the Schrödinger type operator, where $V \neq 0$ is a nonnegative potential and belongs to the reverse Hölder class RH_{q_1} for $q_1 > n/2, n \geq 5$. The higher Riesz transform associated with $\mathcal{L}_2$ is denoted by $\mathcal{R} = \nabla^2 \mathcal{L}_2^{-\frac{1}{2}}$ and its dual is denoted by $\mathcal{R}^* = \mathcal{L}_2^{-\frac{1}{2}} \nabla^2$. In this paper, we consider the m-order commutators $[b^m, \mathcal{R}]$ and $[b^m, \mathcal{R}^*]$, and establish the (L^p, L^q)-boundedness of these commutators when b belongs to the new Campanato space $\Lambda_\beta^\theta(\rho)$ and $1/q = 1/p - m\beta/n$.

Key Words: Schrödinger operator, Campanato space, Riesz transform, commutator.

AMS Subject Classifications: 42B25, 35J10, 42B35

1 Introduction

In this paper, we consider the Schrödinger type operator

$$\mathcal{L}_2 = (-\Delta)^2 + V^2 \quad \text{on } \mathbb{R}^n, \quad n \geq 5,$$

where V is nonnegative, $V \neq 0$, and belongs to the reverse Hölder class RH_q for some $q \geq n/2$, i.e., there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B V(y)^q dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) dy$$

for every ball $B \subset \mathbb{R}^n$.

The higher Riesz transform associated with $\mathcal{L}_2$ is defined by $\mathcal{R} = \nabla^2 \mathcal{L}_2^{-1/2}$, and its dual is defined by $\mathcal{R}^* = \mathcal{L}_2^{-1/2} \nabla^2$. The L^p-boundedness of the higher Riesz transforms

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have been obtained in [1] by Liu and Dong: Suppose $V \in RH_{q_1}$ with $n/2 < q_1 < n$. Let $1/p_1 = 2/q_1 - 2/n$, $p_1' = p_1/(p_1 - 1)$. If $1 < p < p_1$, then for all $f \in L^p(\mathbb{R}^n)$,

$$\|\mathcal{R}f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

If $p_1' < p < \infty$, then for all $f \in L^p(\mathbb{R}^n)$,

$$\|\mathcal{R}^*f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

As in [2], for a given potential $V \in RH_q$ with $q > n/2$, we define the auxiliary function

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\right\}, \quad x \in \mathbb{R}^n.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

Let $\theta > 0$ and $0 < \beta < 1$, in view of [3], the new Campanato class $\Lambda_\beta^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\beta/n}}\int_{B(x,r)} |b(y) - b_B|dy \le C\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_\beta^\theta(\rho)$, denoted by $[b]_\beta^\theta$, is given by the infimum of the constants in the inequalities above.

Note that if $\theta = 0$, $\Lambda_\beta^\theta(\rho)$ is the classical Campanato space; If $\beta = 0$, $\Lambda_\beta^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [4].

We denote by $\mathcal{K}$ and $\mathcal{K}^*$ the kernels of $\mathcal{R}$ and $\mathcal{R}^*$, respectively. Let b be a locally integrable function, m be a positive integer. The m-order commutators generated by higher Riesz transform and b are defined by

$$[b^m, \mathcal{R}]f(x) = \int_{R^n} \mathcal{K}(x,y)(b(x) - b(y))^m f(y)dy$$

and

$$[b^m, \mathcal{R}^*]f(x) = \int_{R^n} \mathcal{K}^*(x,y)(b(x) - b(y))^m f(y)dy.$$

In this paper, we are interested in the boundedness of $[b^m, \mathcal{R}]$ and $[b^m, \mathcal{R}^*]$ on Lebesgue space when b belongs to the new Campanato class $\Lambda_\beta^\theta(\rho)$. The main result of this paper is as follows.

Theorem 1.1. *Suppose $V \in RH_{q_1}$ with $n/2 < q_1 < n, 1/p_1 = 2/q_1 - 2/n, p_1' = p_1/(p_1 - 1)$. Let $0 < \beta < 1$, and let $b \in \Lambda_\beta^\theta(\rho)$. If $p_1' < p < \infty$, then for all $f \in L^p(\mathbb{R}^n)$,*

$$\|[b^m, \mathcal{R}^*]f\|_{L^q(\mathbb{R}^n)} \le C([b]_\beta^\theta)^m \|f\|_{L^p(\mathbb{R}^n)},$$

where $1/q = 1/p - m\beta/n$.