Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 3, pp. 835-848

DOI: 10.4208/aamm.OA-2019-0329
June 2020

Equivalent a Posteriori Error Estimators for Semilinear Elliptic Equations with Dirac Right-Hand Side

Wenting Mao, Yanping Chen* and Haitao Leng

School of Mathematical Sciences, South China Normal University, Guangzhou 520631, Guangdong, China

Received 12 November 2019; Accepted (in revised version) 14 January 2020

Abstract. In this paper, we consider a semilinear elliptic equation with Dirac right-hand side. An equivalent a posteriori error estimator for the L^s norm is obtained. We note that the a posteriori error estimator can be used to design adaptive finite element algorithms. In the end, some examples are provided to examine the quality of the derived estimator.

AMS subject classifications: 65L10, 65L60, 65M70

Key words: Elliptic equation, Dirac, a posteriori error estimator, semilinear, L^s error estimates.

1 Introduction

Elliptic equations with Dirac right-hand side arise in different applications as, for instance, the electric field generated by a point charge, modeling of acoustic monopoles, transport equations for effluent discharge in acquatic media, etc. It is well known that the solution to this type of problem does not belong to H^1 because of the Dirac right-hand side. The singularity of the problem suggests that refined adequately meshes around the delta support should be used to improve the discrete accuracy.

In [1], the authors gave a posteriori error estimates for elliptic problems with Dirac source terms, and these estimators are efficient and reliable for L^p norm and $W^{1,p}$ norm. Agnelli, Garan and Morin also proved in [2] an equivalent a posteriori error estimator for elliptic problems with Dirac measure terms in weighted spaces H^1_α. Additionally, some a priori error estimates for this kind of problem were obtained in [3,5,6]. In [3], the authors proved that the L^2 norm converges with order h in two dimensions and order $h^{\frac{1}{2}}$ in three dimensions. For [1,6], the authors proved the convergence of almost optimal order in L^2 norm by using graded meshes.

*Corresponding author.
Emails: maowenting126@163.com (W. T. Mao), yanpingchen@scnu.edu.cn (Y. P. Chen), lhtdemail@163.com (H. T. Leng)

In addition to the error analysis for elliptic equations with Dirac right-hand side, there also exists error analysis for PDE-constraints optimal control problems, see [7–9] for more details. A priori and a posteriori error estimates were obtained in [7] and [8] for optimal control problems.

At heart of an adaptive finite element method is a posteriori error estimator which is usually used as an indicator to indicate refinement. The goal is to distribute more mesh nodes in the domain where the singularities exist to save the computational cost. Although adaptive finite element methods have been successfully applied for more than three decades, see [10], the convergence analysis of the entire algorithm is rather recent. For one-dimensional case, the earliest work was made by Babuška [11]. The pioneer work for multi-dimensional case was made by Dörfler [12] for Poissons equation. More recent researches can be found in [13, 14] and references therein.

In this paper, a semilinear elliptic equation with Dirac right-hand side is studied. An efficient and reliable a posteriori error estimator is derived, which can be used to design adaptive finite element algorithm. To the best knowledge of the authors, this result is new. In the end we provide some examples to examine the quality of the derived estimator.

The rest of this paper is arranged as follows: we give the model and the discrete formulation in Section 2. In Section 3, we prove an equivalent a posteriori error estimator for the L^s norm. In Section 4, some examples are provided to examine the quality of the derived estimator.

2 Setting of the problem

In this section we start by laying out some general notations. Let $\Omega \in \mathbb{R}^2$ be a polygonal domain. Here we note that we use the standard notations of the Sobolev spaces. For convenience, we agree on the following abbreviations:

$$\|\cdot\|_{0,\omega} := \|\cdot\|_{L^2(\omega)}, \quad |\cdot|_{s,p,\omega} := |\cdot|_{W^{s,p}(\omega)}, \quad \|\cdot\|_{s,\omega} = \|\cdot\|_{H^s(\omega)},$$

for any $\omega \subset \Omega$, $s > 0$ and $1 \leq p < \infty$. If $\omega = \Omega$, we will withdraw the subscript ω for convenience. Then we have

$$\|\cdot\|_0 := \|\cdot\|_{L^2(\Omega)}, \quad |\cdot|_{s,p} := |\cdot|_{W^{s,p}(\Omega)}, \quad \|\cdot\|_s = \|\cdot\|_{H^s(\Omega)}.$$

In this paper we consider the following problem

$$-\Delta u + d(x,u) = \delta_{x_0} \qquad \text{in } \Omega, \tag{2.1a}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{2.1b}$$

where x_0 denotes the center of the Dirac measure, and δ_{x_0} denotes the Dirac measure at $x_0 \in \Omega$ such that $\mathrm{dis}(x_0,\Gamma) > 0$. For function $d(x,u)$ in (2.1a), we make the following assumption.