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Error Estimates for 1D Asymptotic Models in Coaxial Cables with Non-Homogeneous Cross-Section

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Abstract. This paper is the first contribution towards the rigorous justification of asymptotic 1D models for the time-domain simulation of the propagation of electromagnetic waves in coaxial cables. Our general objective is to derive error estimates between the ″exact″ solution of the full 3D model and the ″approximate″ solution of the 1D model known as the Telegraphist's equation.

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Key words: Coaxial cables, Telegraphist's equation, error estimates, non-homogeneous cross-section.

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1 Introduction

This work is a continuation of a previous article [4] devoted to the asymptotic modeling of electromagnetic waves propagations in a thin co-axial cable. By thin cable, we mean a 3D elongated (infinitely long in this paper) cylindrical domain whose transverse dimensions are small with respect to the considered wavelengths. By co-axial cable, we mean that each transverse cross-section of the cable is not simply connected, which is essential. Of course, as a cable is a thin structure whose transverse dimensions are much smaller than the longitudinal one, one would like to use a simplified 1D model: this is even necessary for the effective efficiency of the computational tool (one wants in particular to avoid using a 3D mesh for the thin cable).

In such a situation, electrical engineers use the well-known Telegraphist's equations for ″perfect″ coaxial cables (homogeneous with circular cross-section), where

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the electric unknowns are reduced to an electric potential $V(x_3,t)$ and an electric current $I(x_3,t)$, where x_3 denotes the abscissa along the cable, t is time and in absence of source

$$\begin{cases} C\dfrac{\partial V}{\partial t} + GV - \dfrac{\partial I}{\partial x_3} = 0, \\ L\dfrac{\partial I}{\partial t} + RI - \dfrac{\partial V}{\partial x_3} = 0, \end{cases} \tag{1.1}$$

where the capacitance C, the inductance L, the conductance G and the resistance R can be expressed it terms of the geometry of the cross-section. In [4], thanks to a formal asymptotic expansions with respect to the small parameter $\delta :=$ diameter of the cable/unit reference length, we derived a simplified 1D effective model under quite general assumptions: the cross section is heterogeneous, slowly variable in the longitudinal direction and possibly made of lossy media (i.e., with electric or magnetic conductivities). To derive this effective model, we considered a family of problems posed in domains that depend on a small geometric parameter $\delta > 0$. Of course, a given cable corresponds to a given value of δ but the effective model will be constructed by an asymptotic analysis in δ.

The resulting model appears as an extension of the Telegraphist's equation (1.1) currently used in the engineering community [3, 8] (in particular, we show that the presence of lossy media induces the apparition of time convolution terms in the limit model). The coefficients of the homogenized model are given explicitly as the solutions of two 2D scalar elliptic problems posed in the cable cross-section. Such models can be used as an efficient tool for the time-domain numerical simulation of the propagation of electromagnetic waves in coaxial cables, which is needed in many industrial applications. In our case, we were motivated by the simulation of non-destructive testing experiments by ultra-sounds [5], where coaxial cables are used for the electric supply process for piezo-electric transducers [9].

The present paper is the first contribution towards the rigorous justification of the results of [4]. More than a simple convergence theorem, the general objective is to derive error estimates (in a sense that will be explained later) between the "exact" solution of the full 3D model and the "approximate" solution of the 1D model. We focus in this first paper on the (model) situation of a perfectly cylindrical cable (invariant under translation in the longitudinal direction) whose cross-section is heterogeneous (constitutive coefficients depend on transverse variables) but made of non-lossy media. A more general situation will be considered in a future work.

The paper is organized as follows. In Section 2, we present the considered model problem and more precisely the family of δ-dependent problems that we wish to analyze. In Section 3, we recall the main results of [4] in the simplified situation considered in this paper. Then we give the main results of this work (Theorem 3.2), that provide various error estimates under the only assumption that the data of the problem (the source terms) are adequately "well-prepared". Finally, in Section 4, we give a detailed proof of Theorem 3.2, that relies on appropriate vector field decompositions, energy estimates and adequate versions of Poincaré-Friedrichs inequalities (see Appendix).