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# A New Higher Order Fractional-Step Method for the Incompressible Navier-Stokes Equations

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**Abstract.** In this paper, we present a rigorous error analysis of a new higher order fractional-step scheme for approximation of the time-dependent Navier-Stokes equations. The main feature of the proposed scheme is twofold. First, it is a two-step scheme in which the incompressibility and nonlinearities are split. Second, this scheme is a linear scheme and is simple to implement. It is shown that the proposed scheme possesses the convergence rate $\mathcal{O}((\Delta t)^{3/2})$ in the discrete $l^2(\mathbf{H}_0^1)\cap l^\infty(\mathbf{L}^2)$-norm for the end-of-step velocity. Two different numerical experiments are presented to confirm the theoretical analysis and the efficiency of the proposed scheme.

**AMS subject classifications**: 65M12, 76W05

**Key words**: Incompressible Navier-Stokes equations, fractional-step method, Crank-Nicolson scheme, temporal errors estimates.

## 1 Introduction

The Navier-Stokes equations are used to describe the flow of a viscous and incompressible fluid, which are governed by the following time-dependent nonlinear problems:

$$\frac{\partial \mathbf{u}}{\partial t}-\mu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}, \tag{1.1a}$$

$$\mathrm{div}\mathbf{u}=0, \tag{1.1b}$$

for $x\in\Omega$ and $t\in(0,T]$ with $T>0$, where $\Omega$ is an open bounded domain in $\mathbf{R}^d$ ($d$=2 or 3) with a sufficiently smooth boundary $\partial\Omega$. The constant $\mu>0$ represents the kinematic viscosity.

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The vector-value function $\mathbf{f}$ represents the body forces applied to the fluid. To study (1.1a)-(1.1b), the appropriate initial and boundary conditions are needed. For the sake of simplicity, in this paper, we consider the following initial and boundary conditions:

$$\mathbf{u}(x,0)=\mathbf{u}_0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{u}=0 \quad \text{on } \partial\Omega\times[0,T]. \tag{1.2}$$

In (1.1a)-(1.2), the unknown are the fluid velocity $\mathbf{u}$ and the fluid kinematic pressure $p$ which are coupled by the incompressible condition $\mathrm{div}\mathbf{u}=0$. This condition is one of the main concern in designing efficient time discrete schemes for numerical simulation of (1.1a)-(1.2). The well-known projection method, initially proposed by Chorin [4] and Temam [22], is designed to decouple the velocity and pressure, and has been further developed in various directions [12,19–21]. This method is first to compute an intermediate velocity field without taking into account incompressibility and then perform a pressure correction, which is a projection back to the subspace of solenoidal (divergence-free) vector field. Shen proved that this projection method is first order accurate in the time step size [19]. The incompatibility of the projection boundary conditions with (1.2) may result in a numerical boundary layer of size $\mathcal{O}(\sqrt{\Delta t})$, where $\Delta t$ is the time step size [18,24].

The viscosity-splitting fractional-step method, proposed by Blasco-Codina-Huerta [2], also is an efficient algorithm for numerical simulation of the incompressible Navier-Stokes equations. It is a two-step scheme in which the incompressibility and nonlinearities of the Navier-Stokes problems are split into different steps and allows the enforcement of the original boundary conditions in all substeps. It was shown that the intermediate and end-of-step velocities converge to a continuous solution in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ [2]. Moreover, these velocities and pressure were first-order accurate in the time step size [3, 10]. Subsequently, Dai studied a nonlinear higher order viscosity splitting, fractional-step scheme [5]. However, one has to solve a nonlinear problem at each time step, which results in a time-consuming in the practical computations. Recently, the first-order viscosity-splitting fractional-step methods have been applied to other nonlinear partial differential systems, such as the three-dimensional incompressible MHD systems [1] and the primitive equations in the field of geophysical fluids [11].

In this paper, based upon Crank-Nicolson discretization scheme in time, we will study a higher order fractional-step scheme for the approximation of (1.1a)-(1.2). Unlike the nonlinear scheme in [5], the proposed scheme is a semi-implicit scheme and it only solves two linear systems at every time-step. Therefore, it is simple to implement. To state main result, we introduce the following notations. Let $X$ be a Banach space equipped with norm $\|\cdot\|_X$. Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform partition of time interval $[0,T]$ with the time step size $\Delta t=T/N$ and $t_n=n\Delta t$ for $0\le n\le N$. We denote two discrete norms by

$$\|\mathbf{u}^n\|_{l^2(X)}=\left(\Delta t\sum_{n=1}^{N}\|\mathbf{u}^n\|_X^2\right)^{1/2}, \qquad \|\mathbf{u}^n\|_{l^\infty(X)}=\max_{1\le n\le N}\|\mathbf{u}^n\|_X.$$

In this paper we will show that the proposed higher order fractional-step scheme provides the temporal error estimates of $\mathcal{O}((\Delta t)^{3/2})$ for the end-of-step velocity in $l^2(\mathbf{H}_0^1)\cap$