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A Note on Existence of a Bound State for a Non-Autonomous Nonlinear Scalar Field Equation

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Abstract. The aim of this paper is to present a positive solution of a semilinear elliptic equation in $\mathbb{R}^N$ with non-autonomous non-linearities which are not necessarily pure-powers, nor homogeneous, and which are superlinear or asymptotically linear at infinity. The proof is variational combined with topological arguments.

Key Words: Schrödinger equation, asymptotically linear, superlinear, variational methods.

AMS Subject Classifications: 35Q55, 35B09, 35J20

1 Introduction

Semilinear elliptic equations in $\mathbb{R}^N$ arise as stationary states of Schrödinger or Klein-Gordon type equations, when modelling, for instance, the propagation of a light beam in Kerr and non-Kerr media, see [2,25] and references therein, leading to the problem

$$\begin{cases} -\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{P}$$

The search for solutions of nonlinear scalar field equations using variational methods has been intensive in the past three decades, see [6,8,9,13,22,24], among many others.

The interest in this kind of problem is twofold: on one hand the large range of applications and on the other hand the mathematical challenge introduced when working in an unbounded domain like $\mathbb{R}^N$.

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In this work we are mainly concerned with the following simplified version of problem (P):

$$\begin{cases} -\Delta u + u = (1+a(x))f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_a}$$

with assumptions on $a(x)$ which imply that this problem may not have a least energy solution and it is a challenge to look for solutions in higher energy levels. Our special motivation was the notable paper of Bahri and Li [5] where they introduced a min-max procedure to prove the existence of a positive bound state solution of

$$\begin{cases} -\Delta u + u = q(x)|u|^{p-1}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_q}$$

where $1 < p < \frac{N+2}{N-2} = 2^* - 1$, if $N \geq 3$, and $1 < p < +\infty$, if $N \in \{1,2\}$ and $q \in L^\infty(\mathbb{R}^N)$ satisfying some exponential asymptotic limit, when a ground state does not exist for the problem.

Our objective is to extend [5] to non homogeneous non-linearities f which are either superlinear or asymptotically linear at infinity and $a(x)$ also satisfying an exponential asymptotic limit. We use a variational approach and a topological argument introduced in [5] and updated in [12,14,19].

There is an extensive literature on this subject. We are going to highlight some articles which are more relevant with respect to our main objectives. In the autonomos cases where $V(x) = m$ and $f(x,u) = f(u)$, the pioneering work of Berestycki and Lions [9] exhibited a ground state solution for (P). Using constrained minimization arguments, they showed the existence of a positive, radial solution and investigated its regularity and its exponential decay at infinity. In 1984, P. L. Lions [18] introduced breakthrough ideas of concentration-compactness that enabled numerous investigations on this subject matter.

Lehrer and Maia [17] studied problem (P) with $V(x) = \lambda > 0$ and $f(x,u) = a(x)f(u)$ in $\mathbb{R}^N$, asymptotically linear at infinity, and imposed several conditions on $a(x)$. Working in a so-called Pohozaev manifold and using a linking argument they proved existence of a bound state solution of the problem. In our work, we want to attenuate the restrictions on $a(x)$.

Clapp and Maia [12] established existence of a positive solution to the stationary nonlinear Schrödinger equation $-\Delta u + V(x)u = f(u)$ in $\mathbb{R}^N$ where f is either superlinear or asymptotically linear at infinity using variational techniques including the case where the critical level of minimal energy is not attained. Our result is a counterpart of this together with improvent on the hypotheses.

Recently, Weth and Evequoz [14] considered the equation (P) under assumptions on $a(x)$, which led them to work with the space $H^1(\mathbb{R}^N)$ under a spectral decomposition $E^+ \oplus E^0 \oplus E_-$ and with F, the primitive of f, of superquadratic type at infinity. In order to