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Regularized Interpolation Driven by Total Variation

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Abstract. We explore minimization problems of the form

$$\operatorname{Inf}\left\{\int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \alpha \int_0^1 |u|^2\right\},$$

where u is a function defined on $(0,1)$, (a_i) are k given points in $(0,1)$, with $k \geq 2$, (f_i) are k given real numbers, and $\alpha \geq 0$ is a parameter taken to be 0 or 1 for simplicity. The natural functional setting is the Sobolev space $W^{1,1}(0,1)$. When $\alpha = 0$ the Inf is achieved in $W^{1,1}(0,1)$. However, when $\alpha = 1$, minimizers need not exist in $W^{1,1}(0,1)$. One is led to introduce a relaxed functional defined on the space $BV(0,1)$, whose minimizers always exist and can be viewed as generalized solutions of the original ill-posed problem.

Key Words: Interpolation, minimization problems, functions of bounded variation, relaxed functional.

AMS Subject Classifications: 26B30, 49J45, 65D05

1 Introduction

Given k points, with $k \geq 2$,

$$0 < a_1 < a_2 < \cdots < a_k < 1, \tag{1.1}$$

and k real numbers f_i, $i = 1, \cdots, k$, the aim is to find a function u defined on $(0,1)$ such that $u(a_i)$ approximates f_i as best as possible, and keeping at the same time some control on the regularity of u, measured here in terms of total variation of u. For this purpose define the functional

$$F(u) = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2, \tag{1.2}$$

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and then minimize F. (One may also insert a fidelity parameter in front of the first integral, but we take to be 1 for simplicity). Note that F is well-defined on the Sobolev space $W^{1,1}(0,1)$ since $W^{1,1}(0,1) \subset C([0,1])$, so that $u(a_i)$ makes sense. As is well-known $W^{1,1}(0,1)$ is not a good function space from the point of view of minimization techniques in Functional Analysis. Often, variational problems do *not* admit minimizers in $W^{1,1}(0,1)$. To make up for this "defect" one is usually led to enlarge $W^{1,1}(0,1)$ and replace it by $BV(0,1)$, the space of functions of bounded variation (see e.g., [1,2,5]), where the existence of minimizers is often a matter of routine. The drawback is that the specific functional F is not properly defined on $BV(0,1)$ since the term $u(a_i)$ has no obvious meaning when u has a jump at a_i.

In Section 2 we establish that (surprisingly!) the problem

$$\operatorname*{Inf}_{u\in W^{1,1}(0,1)} F(u) \tag{1.3}$$

always admits minimizers. In fact all minimizers are classified with the help of a finite-dimensional auxiliary problem. Given

$$\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k,$$

set

$$\Phi(\lambda) := \sum_{i=1}^{k-1} |\lambda_{i+1} - \lambda_i| + \sum_{i=1}^{k} |\lambda_i - f_i|^2. \tag{1.4}$$

By convexity

$$m := \min_{\lambda\in\mathbb{R}^k} \Phi(\lambda) \tag{1.5}$$

is achieved by some unique λ denoted

$$U = (U_1, \cdots, U_k),$$

and which plays an important role throughout the paper. In this section we never invoke Functional Analysis and the space $BV(0,1)$ is noticeably absent. The existence of minimizers in $W^{1,1}(0,1)$ is derived from an *elementary* computation originally due to T. Sznigir [6,7]. However this "miracle" does not repeat itself: as we are going to see in Section 5 even "mild" pertubations of F need not admit minimizers in $W^{1,1}(0,1)$, and there it will be essential to "relax" the problem and search for minimizers in $BV(0,1)$ using tools of Functional Analysis.

In Section 3 we introduce the relaxed functional F_r of F, which is much better suited to minimization problems involving the functional F. We start with the standard abstract formulation, namely F_r is defined for every $v \in BV(0,1)$ by

$$F_r(v) := \operatorname{Inf} \liminf_{n\to\infty} F(v_n), \tag{1.6}$$

where the Inf in (1.6) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$. The main result, Theorem 3.1, provides an *explicit* formula for F_r. The major