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A New C-Eigenvalue Localisation Set for Piezoelectric-Type Tensors

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Abstract. A new inclusion set for localisation of the C-eigenvalues of piezoelectric tensors is established. Numerical experiments show that it is better or comparable to the methods known in literature.

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Key words: C-eigenvalue, C-eigenvector, piezoelectric tensor, C-eigenvalue localisation theorem.

1. Introduction

Third order tensors play an important role in physics and engineering, including nonlinear optics [10,12], properties of crystals [6,11,19,20,22,26] and liquid crystals [5,9,24]. In particular, piezoelectric tensors find wide applications in converse piezoelectric and piezoelectric effects [4]. Chen *et al.* [4] specify the piezoelectric-type tensors as follows.

Definition 1.1 (cf. Chen *et al.* [4])**.** A third order n-dimensional tensor $\mathscr{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ is called the piezoelectric-type tensor if the last two indices of $\mathscr{A}$ are symmetric — i.e. if $a_{ijk} = a_{ikj}$ for all $j,k \in [n]$, where $[n] := \{1,2,\dots,n\}$.

Qi [21] and Lim [18] introduced the notion of eigenvalues for higher order tensors. It is worth noting that the eigenvalues of the third order symmetric traceless-tensors are widely used in the theory of liquid crystals [5,9,24]. Following these ideas, Chen *et al.* [4] defined C-eigenvalues and C-eigenvectors for piezoelectric-type tensors, which turn out to be useful in the study of piezoelectric and converse piezoelectric effects in solid crystals.

Definition 1.2 (cf. Chen *et al.* [4])**.** Let $\mathscr{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a third-order n-dimensional tensor. A number $\lambda \in \mathbb{R}$ is called the C-eigenvalue of $\mathscr{A}$ if there are $x, y \in \mathbb{R}^n$ such that

$$\mathscr{A}yy = \lambda x, \quad x\mathscr{A}y = \lambda y, \quad x^\top x = 1, \quad y^\top y = 1, \tag{1.1}$$

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where

$$(\mathscr{A}yy)_i = \sum_{k,j\in[n]} c_{ikj}y_ky_j, \quad (x\mathscr{A}y)_i = \sum_{k,j\in[n]} c_{kji}x_ky_j.$$

The vectors x and y are referred to as associated left and right C-eigenvectors, respectively.

By $\sigma(\mathscr{A})$ we denote the C-spectrum of the piezoelectric-type tensor $\mathscr{A}$ — i.e. the set of all C-eigenvalues of the piezoelectric-type tensor $\mathscr{A}$. The C-spectral radius of $\mathscr{A}$ is defined by

$$\rho(\mathscr{A}) := \max\{|\lambda| : \lambda \in \sigma(\mathscr{A})\}.$$

For a piezoelectric tensor $\mathscr{A}$, Chen *et al.* [4] proved the existence of C-eigenvalues associated with left and right C-eigenvectors. They also showed that the largest C-eigenvalue of the piezoelectric tensor represents the highest piezoelectric coupling constant and it can be determined as

$$\lambda^* = \max\left\{x\mathscr{A}yy : x^\top x = 1,\, y^\top y = 1\right\},$$

where

$$x\mathscr{A}yy := \sum_{i,k,j\in[n]} c_{ijk}x_iy_jy_k.$$

However, the practical calculation of λ^* is a challenging problem because of the uncertainty with the C-eigenvectors x and y in actual operations. On the other hand, we can capture all eigenvalues of a high order tensor by the eigenvalue localisation. In particular, for real symmetric tensors, Qi [21] considers an eigenvalue localisation set, which is an extension of the Geršgorin matrix eigenvalue inclusion theorem for matrices [23]. For general tensors, Li *et al.* [16] proposed Brauer-type eigenvalue inclusion sets. Later on, various eigenvalue localisation sets and their applications have been studied in Refs. [1,2,8,13,14,17,25,27].

Recently, C. Li and Y. Li [15] introduced two intervals to estimate all C-eigenvalues of a piezoelectric-type tensor.

Theorem 1.1 (cf. C. Li & Y. Li [15]). *If λ is a C-eigenvalue of the piezoelectric-type tensor $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$, then*

$$\lambda \in [-\rho, \rho],$$

where

$$\rho = \max_{i,j\in[n]} \left\{R_i^{(1)}(\mathscr{C})R_j(\mathscr{C})\right\}^{1/2},$$

$$R_i^{(1)}(\mathscr{C}) = \sum_{l,k\in[n]} |c_{ilk}|, R_j(\mathscr{C}) = \sum_{l,k\in[n]} |c_{lkj}|, \quad [n] = \{1,2,\ldots,n\}.$$

Theorem 1.2 (cf. C. Li & Y. Li [15]). *If λ is a C-eigenvalue of the piezoelectric-type tensor $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$ and S is a subset of $[n]$, then*

$$\lambda \in \left[-\rho_s, \rho_s\right],$$