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Numerical Analysis of an Adhesive Contact Problem with Long Memory

Xiaoliang Cheng and Qichang Xiao*

School of Mathematical Sciences, Zhejiang University,
Hangzhou 310027, P.R. China.

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Abstract. Spatially semidiscrete and fully discrete schemes for a variational-hemivariational inequality, which describes adhesive contact between a deformable body of a viscoelastic material with long memory and a foundation are constructed. The variational formulation of the problem is represented by a system coupling a nonlinear integral equation with a history-dependent variational-hemivariational inequality. Assuming certain regularity of the solution and using piecewise linear finite element function for displacements and piecewise constant functions for bonding field, we obtain optimal order error estimates.

AMS subject classifications: 65M10, 78A48

Key words: Variational-hemivariational inequality, adhesion, memory term, numerical approximation, error estimate.

1. Introduction

The mathematical theory of hemivariational inequalities plays an important role in a variety of subjects, ranging from nonsmooth mechanics, physics, and engineering to economics. As the result, a large number of mechanical contact problems lead to mathematical models expressed in terms of hemivariational inequalities. In recent years, the mathematical theory devoted to this field grows rapidly. Several results on hemivariational inequalities can be found in [9, 13–15].

In spite of vast literature on the modeling of contact problems and associated hemivariational inequalities, numerical methods for the problems with adhesion are not well developed. In this work, we deal with the numerical analysis of an adhesive contact problem for viscoelastic materials with long memory. Such models are closely connected to variational-hemivariational inequalities.

Let us recall the contact problem studied in [7]. Suppose that a viscoelastic body occupies an open bounded connected set Ω in the space $\mathbb{R}^d$, where $d = 2$ or $d = 3$. The Lipschitz

*Corresponding author. *Email addresses:* xiaoliangcheng@zju.edu.cn (X. Cheng), xiao_qc@126.com (Q. Xiao)

continuous boundary Γ of Ω consists of three mutually disjoint measurable sets Γ_D, Γ_N and Γ_C such that the $(d-1)$-dimensional Hausdorff measure of Γ_D is positive. We are interested in the evolution of the mechanical state of the body on a finite time interval $[0,T]$. This evolution is caused by volume forces of density $\boldsymbol{f}_0$ in Ω and by surface tractions of density $\boldsymbol{f}_N$ on Γ_N. Besides, the viscoelastic body is in contact with another body called foundation or obstacle over the surface Γ_C. The surface Γ_C is referred to as the contact surface. We introduce the following notation. Let $\mathbb{S}^d = \mathbb{R}^{d\times d}_{sym}$ be a space of symmetric $d\times d$ real matrices and

$$Q=\Omega\times(0,T),\quad \Sigma_D=\Gamma_D\times(0,T),\quad \Sigma_N=\Gamma_N\times(0,T),\quad \Sigma_C=\Gamma_C\times(0,T).$$

The classical formulation of the problem is as follows.

Problem 1.1. Find a displacement field $\boldsymbol{u}: Q\to\mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}: Q\to\mathbb{S}^d$ such that

$$\boldsymbol{\sigma}(t)=\mathscr{A}(t,\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))+\int_0^t \mathscr{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t)+\boldsymbol{f}_0(t)=\boldsymbol{0} \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}(t)=\boldsymbol{0} \qquad \text{on } \Sigma_D, \tag{1.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu}=\boldsymbol{f}_N(t) \qquad \text{on } \Sigma_N, \tag{1.4}$$

$$\begin{cases} u_\nu(t)\le g, \quad \sigma_\nu(t)+p_\nu(\beta(t),u_\nu(t))\le 0,\\ \big(u_\nu(t)-g\big)\big(\sigma_\nu(t)+p_\nu(\beta(t),u_\nu(t))\big)=0 \end{cases} \qquad \text{on } \Sigma_C, \tag{1.5}$$

$$-\boldsymbol{\sigma}_\tau(t)\in\partial j_\tau(\beta(t),\boldsymbol{u}_\tau(t)) \qquad \text{on } \Sigma_C, \tag{1.6}$$

$$\dot\beta(t)=F(u_\nu(t),\boldsymbol{u}_\tau(t),\beta(t)) \qquad \text{on } \Sigma_C, \tag{1.7}$$

$$\beta(0)=\beta_0 \qquad \text{on } \Sigma_C \tag{1.8}$$

for all $t\in(0,T)$.

Here and in what follows, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearised (small) strain tensor with components $\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})=(\boldsymbol{\varepsilon}(\boldsymbol{u}))_{ij}=(u_{i,j}+u_{j,i})/2$ and $u_{i,j}=\partial u_i/\partial x_j$. Moreover, $\operatorname{Div}\boldsymbol{\sigma}=(\sum_{j=1}^d\sigma_{ij,j})$ stands for the divergence of the stress tensor $\boldsymbol{\sigma}$ and the dot above a variable represents time derivative. For simplicity, we do not show the dependence of various functions on variables $\boldsymbol{x}$ and t explicitly. The Eq. (1.1) represents the viscoelastic constitutive law with an elasticity operator $\mathscr{A}$ and a relaxation operator $\mathscr{R}$, (1.2) is the equilibrium equation and (1.3) and (1.4) are the displacement and the traction boundary conditions, respectively.

The contact condition (1.5) without adhesion was first introduced in [10]. It shows that the contact follows a normal compliance condition with the adhesion

$$\sigma_\nu(t)=-p_\nu(\beta(t),u_\nu(t))$$

up to the limit g and if the limit is reached, the contact follows the Signorini-type unilateral condition with the gap g.