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Shape Analysis and Solution to a Class of Nonlinear Wave Equation with Cubic Term

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Abstract. In this paper, we analyze the relation between the shape of the bounded traveling wave solutions and dissipation coefficient of nonlinear wave equation with cubic term by the theory and method of planar dynamical systems. Two critical values which can characterize the scale of dissipation effect are obtained. If dissipation effect is not less than a certain critical value, the traveling wave solutions appear as kink profile; while if it is less than this critical value, they appear as damped oscillatory. All expressions of bounded traveling wave solutions are presented, including exact expressions of bell and kink profile solitary wave solutions, as well as approximate expressions of damped oscillatory solutions. For approximate damped oscillatory solution, using homogenization principle, we give its error estimate by establishing the integral equation which reflects the relations between the exact and approximate solutions. It can be seen that the error is an infinitesimal decreasing in the exponential form.

AMS subject classifications: 35Q51, 37C29

Key words: Nonlinear wave equation, planar dynamical system, exact solutions, approximate damped oscillatory solutions, error estimate.

1 Introduction

The Klein-Gordon equation with cubic nonlinear term

$$u_{tt} - \beta u_{xx} + a_1 u + a_3 u^3 = 0 \tag{1.1}$$

is the important equation of motion of a quantum scalar or pseudoscalar field. Many researchers have studied Eq. (1.1) in recent years. Wazwaz [1] used the tanh method

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to obtain traveling wave solutions with compactons, solitons, solitary patterns and periodic structures of Eq. (1.1); Jang [2] used the auxiliary equation to study Eq. (1.1) and presented some exact traveling wave solutions; Sirendaoreji gave us new and more traveling wave solutions of Eq. (1.1) in [3]; Ye obtained all explicit expressions of the bounded traveling wave solutions for Eq. (1.1) in [4].

As far as we know, dissipation is inevitable in practical problem. So it is essential to study the following equation

$$u_{tt} - \beta u_{xx} + r u_t + a_1 u + a_3 u^3 = 0. \tag{1.2}$$

Eq. (1.2) is a vital wave equation for nonlinear researchers. It can be regarded as the nonlinear telegraph equation [5], which was proposed when the electric cables were paved at the bottom of the Atlantic. It is very useful in telegraph signal transmission. Additionally, it also can be used to describe the pressure produced by pulsation when the blood flows in arterials.

Comparing with Eq. (1.1), Eq. (1.2) is much more complex. The effect of the dissipation term ru_t makes the shape of the traveling wave solutions for Eq. (1.2) varies a lot. Fan [6] and Shang [7] gave two kink profile solitary wave solutions of Eq. (1.2) by homogeneous balance method and Wu-eliminition method; and a kind of direct combination and ansatze method, respectively. We can prove that the solutions obtained by them are equivalent. Zhang [8] offered us a method to judge the shape of solitary wave and some solitary wave solutions of Eq. (1.2). However, the relation between the shape and the dissipation coefficient or the damped oscillatory solutions was not obtained in [8]. Ma [9] considered the explicit traveling wave solutions of Kolmogorov-Petrovskii-Piskunov equation [10]

$$u_{xx} - u_t + a_1 u + a_2 u^2 + a_3 u^3 = 0, \tag{1.3}$$

and described nonlinear interactions of traveling waves by Cole-Hope transformation.

From the above studies, we may ask the following questions. How many bounded traveling wave solutions does Eq. (1.2) have? Are there any bounded solutions have not been obtained? How to express the solutions which can not be found out easily? We are sure that if these questions were solved, it can help us settle many practical problems, such as helping the engineers to control the telegraphic signals. In this paper, we will answer these questions. We firstly apply the theory and method of planar dynamical systems to make qualitative analysis to the dynamical systems which the traveling wave solutions of Eq. (1.2) corresponds to, present the global phase portraits, and study the number and shapes of the bounded traveling wave solutions. Secondly, we analyze the dissipation effect on the shape of bounded traveling wave solutions, and give two critical values which can characterize the scale of the dissipation effect, i.e., if r^2 is not less than a certain critical value, the traveling wave solutions of Eq. (1.2) appear as kink profile; if r^2 is less than this critical value, they appear as damped oscillatory. Based on above analysis, we give the expressions of the bounded traveling wave solutions for Eq. (1.2), including the exact expressions of bell and kink profile