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A New Post-Processing Technique for Finite Element Methods with L^2-Superconvergence

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Abstract. A simple post-processing technique for finite element methods with L^2-superconvergence is proposed. It provides more accurate approximations for solutions of two- and three-dimensional systems of partial differential equations. Approximate solutions can be constructed locally by using finite element approximations u_h provided that u_h is superconvergent for a locally defined projection $\widetilde{P}_h u$. The construction is based on the least-squares fitting algorithm and local L^2-projections. Error estimates are derived and numerical examples illustrate the effectiveness of this approach for finite element methods.

AMS subject classifications: 65N30, 65N15

Key words: Finite element method, post-processing, least-square fitting, L^2-superconvergence.

1. Introduction

The post-processing of approximate solutions is a commonly used procedure to obtain more accurate approximations for important quantities in numerical methods for partial differential equations [4–6, 22, 23]. Post-processing or/and recovery techniques have been developed for plenty of finite element methods with superconvergence [1, 7, 8, 10, 12, 13, 15, 18, 20]. In particular, for the Raviart-Thomas and Brezzi-Douglas-Marini mixed elements methods for second order elliptic problems, the post-processed approximations with improved accuracy are constructed via element-by-element solution of local problems with respect to the finite element solutions of the scalar variable and the Lagrange multiplier [1, 8]. In contrast to the post-processing methods [1, 8], Stenberg [18] proposed an approach based on solving local problems with respect to the mixed finite element approximations of the scalar variable and its gradient. Following ideas of [18], Cockburn *et al.* [12, 13] developed an element-by-element post-processing of the scalar variable for the elliptic problems and velocity variable in the Stokes problem for HDG methods.

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Bramble and Xu [7] proposed a general post-processing technique for various mixed finite element methods with the superconvergence estimate

$$\|\widetilde{P}_h u - u_h\|_{L^p(\Omega)} \leq Ch^{k+2}|\log h|^{\mu_1} \tag{1.1}$$

and the gradient approximation estimate

$$\|\nabla u - (\nabla u)_h\|_{L^p(\Omega)} \leq Ch^{k+1}|\log h|^{\mu_2},$$

where u is the exact solution of a system of partial differential equations on a domain $\Omega \subset \Re^2$, $C > 0$ a generic constant, which depends on u but not on the mesh size h; μ_1, μ_2 are nonnegative constants and $u_h \in W_h$ and $(\nabla u)_h \in V_h$ are finite element approximations of u and ∇u, respectively. Moreover, W_h and V_h are finite-dimensional subspaces of $L^p(\Omega)$, $p \geq 1$, W_h consists of discontinuous piecewise polynomials of degree at most $k \geq 0$, and $\widetilde{P}_h$ is a locally defined operator, which is invariant on polynomials of degree k. Under a regularity condition for u, the post-processed approximation u_h^* obtained from u_h and $(\nabla u)_h$, satisfies the estimate

$$\|u - u_h^*\|_{L^p(\Omega)} \leq C\left(\|\widetilde{P}_h u - u_h\|_{L^p(\Omega)} + h\|\nabla u - (\nabla u)_h\|_{L^p(\Omega)} + h^{k+2}\right).$$

Further, Zienkiewicz and Zhu [22, 23] used the well-known gradient recovery technique, usually referred to as superconvergence patch recovery (SPR), to post-process the gradient ∇u_h of the finite element solution u_h. They constructed an SPR-recovered gradient by a local discrete least-squares fitting of polynomials of degree k to the gradient values at sampling points on element patches. The superconvergence properties of this technique was discussed in Refs. [14,19,21]. Zhang and Naga [20] introduced a different gradient recovery method called the polynomial preserving recovery (PPR). To determine a recovered gradient, the method uses the least-squares algorithm to assign a polynomial of degree $k+1$ to the solution at chosen nodal points and computes the corresponding partial derivatives. Under certain conditions, the PPR post-processed gradient $G_h u_h$ satisfies the superconvergence estimate

$$\|\nabla u - G_h u_h\|_{L^\infty(\Omega_0)} \leq C\left(h^{k+1}|\log h|^{\bar{r}} + h^{k+\sigma}\right),$$

where σ is a positive constant, $\Omega_0 \subset\subset \Omega$, $\bar{r} = 1$ if $k = 1$ and $\bar{r} = 0$ if $k \geq 2$.

However, to the best of authors' knowledge, there is no post-processing technique, which uses only u_h to construct a superconvergent post-processed approximation u_h^*. Here, we present a general post-processing technique for direct construction of the improved approximation of u. The method is based on the least-squares algorithm and the local L^2-projection to determine a fitting polynomial from the finite element solution u_h. Our analysis depends only on a superconvergence result similar to (1.1) and the main result is proved in general approximation-theoretic settings. Therefore, its application is not restricted to the above mentioned finite element methods.

The rest of the paper is organised as follows. Section 2 contains necessary notations. Section 3 is devoted to the construction of the post-processed approximation, the error estimation, and the verification of assumptions. Finally, numerical results in Section 4 are aimed to verify the performance of the post-processing method proposed.