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# A Mixed Formulation of Stabilized Nonconforming Finite Element Method for Linear Elasticity

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**Abstract.** Based on the primal mixed variational formulation, a stabilized nonconforming mixed finite element method is proposed for the linear elasticity problem by adding the jump penalty term for the displacement. Here we use the piecewise constant space for stress and the Crouzeix-Raviart element space for displacement. The mixed method is locking-free, i.e., the convergence does not deteriorate in the nearly incompressible or incompressible case. The optimal convergence order is shown in the $L^2$-norm for stress and in the broken $H^1$-norm and $L^2$-norm for displacement, respectively. Finally, some numerical results are given to demonstrate the optimal convergence and stability of the mixed method.

**AMS subject classifications**: 65N15, 65N30

**Key words**: Mixed method, nonconforming finite element, elasticity, locking-free, stabilization.

## 1 Introduction

Different variational formulations of the linear elasticity problem have been developed in the past. In the sense of weak form, it is possible to show the equivalence to the classical formulation, which consists of the constitutive equation and the equilibrium equation and is well-defined for differentiable stress and displacement. A very common formulation is the pure displacement formulation, where only the displacement is sought in $H^1$-space. Therein, the stress tensor is eliminated by using the constitutive equation, as it can be expressed in terms of the displacement for finite $\lambda$ where $\lambda$ is the Lamé constant. However, when this formulation is used, one may expect the stability problem in case of nearly incompressible or incompressible materials, such that the locking phenomenon may appear. This is due to the fact that, the compliance tensor, which links

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the stress and strain, becomes singular in the incompressible limit (the case $\lambda=\infty$), such that the elasticity tensor, the inverse of the compliance tensor acting as a coefficient in the pure displacement formulation, deteriorates with the growing incompressibility of material ($\lambda\to\infty$). In the incompressible limit, the inverse of compliance tensor doesn't exist, so the stress can not be expressed in terms of the displacement, such that it is not possible to pose the pure displacement formulation in this case. For this formulation, some locking-free finite element methods have been developed where the convergence is uniform with respect to the Lamé constant $\lambda$, see, e.g., [9,13,21,30,31,40], which will be further discussed in the following.

Relatively, the mixed variational formulations are preferable to the pure displacement formulation for modeling of nearly incompressible or incompressible materials, where both the stress and displacement are simultaneously considered as unknowns as they show better stability properties. Moreover, the stress is usually a physical quantity of primary interest. Although it can be obtained in the pure displacement method by differentiating the displacement, but this may degrade the order of the approximation.

Here we consider the Hellinger-Reissner mixed formulation. There are essentially two possibilities to apply the derivatives to the displacement or the stress. The first one is the primal mixed variational formulation, which is easy to discretize but leads usually to the standard methods suffering from locking unless special techniques are applied. Before surveying further the first one, we browse the second one called the dual mixed variational formulation. The second one needs stress tensor elements with continuous normal components, which is very difficult to construct due to the symmetry and stability requirements from this formulation, but may lead to better approximation properties. While such stable conforming element pairs have been successfully constructed in both two and three dimensions, the resulting stress elements tend to be quite complicated, especially in three dimensions, see e.g., [1,2,5,6,27]. We also mention the further development of stable conforming elements from the references [12,22,24,25], where a new class of stable conforming elements called the Hu-Zhang element is proposed. On the other hand, much attention has been paid to constructing the nonconforming mixed finite elements, which relax the interelement continuity requirement and seem to be simpler, see [3,7,23] and the references therein. However, the convergence of nonconforming mixed finite element methods has not been considered for nearly incompressible or incompressible materials in the works mentioned above, where the compliance tensor is only assumed to be bounded and symmetric positive definite. Besides, we mention that, some mixed elements with weaker symmetry have been developed and can be shown to be uniformly convergent with respect to the Lamé constant $\lambda$, see, e.g., [4,29,32,35].

Now we turn back to the primal mixed variational formulation. As mentioned before, for finite $\lambda$ this formulation can be reduced to the pure displacement formulation by eliminating the stress. By splitting the elasticity operator into the gradient operator and divergence operator with appropriate coefficients, Brenner and Sung [9] used the Crouzeix-Raviart (CR) element [17] to develop a locking-free nonconforming finite element method for the pure displacement problem of nearly incompressible elasticity.