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Discontinuous Galerkin Methods for Multi-Pantograph Delay Differential Equations

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Abstract. In this paper, the discontinuous Galerkin method is applied to solve the multi-pantograph delay differential equations. We analyze the optimal global convergence and local superconvergence for smooth solutions under uniform meshes. Due to the initial singularity of the forcing term f, solutions of multi-pantograph delay differential equations are singular. We obtain the relevant global convergence and local superconvergence for weakly singular solutions under graded meshes. The numerical examples are provided to illustrate our theoretical results.

AMS subject classifications: 65L60, 65L70

Key words: Multi-pantograph, discontinuous Galerkin method, global convergence, local superconvergence, weakly singular, graded meshes.

1 Introduction

This paper deals with the properties of the following linear multi-pantograph delay differential equation (MPDDE),

$$u'(t)=a(t)u(t)+\sum_{i=1}^{l}b_i(t)u(q_it)+f(t), \qquad t\in J:=[0,T], \tag{1.1a}$$

$$u(0)=u_0, \tag{1.1b}$$

where $a(t),b_i(t)$ are continuous functions, $q_i\in(0,1)$, $(i=1,2,\cdots,l)$ are delay coefficients.

As one of the most important mathematical models, MPDDE is widely used in many fields such as engineering, biology systems, physics and medicine. The study of the MPDDE has been a rapid development by many authors numerically and analytically

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these years. Ishiwata [1] analyzed the attainable order of collocation method for neutral functional-differential equations with proportional delays. Li and Liu [2] used the Runge-Kutta method to solve the multi-pantograph delay differential equation. Taylor method was also used to solve multi-pantograph delay differential equations, such as the paper by Sezer et al. [3]. Brunner [4] applied the collocation method to the pantograph-type Volterra functional equation with multiple delays and Yu [5] used the variational iteration method to solve the multi-pantograph delay differential equation, respectively. Feng [6] employed the homotopy perturbation method to solve multi-pantograph delay differential equations with variable coefficients. Lately, Geng and Qian [7] solved the singularly perturbed multi-pantograph delay differential equations based on the reproducing kernel space method. Komashynska et al. [8] used the residual power series method to solve a system of multi-pantograph delay differential equations. Davaeifar and Rashidinia [9] utilized collocation methods for a system of multi-pantograph type delay differential equations with variable coefficients and obtained the approximate solutions based on the Boubaker polynomials. Zheng et al. [10] developed a Legendre-collocation spectral method for the second order Volterra integro-differential equation with delay. Sedaghat et al. [11] provided a spectral method based on the operational matrices of the Legendre polynomials to solve neutral multi-pantograph delay differential equations.

The discontinuous Galerkin (DG) method was first proposed in [12] as a nonstandard finite element method for numerical solutions of neuron transport problems. Then DG methods are extensively used in solving partial differential equations and integral differential equations. DG methods are also successfully applied to delay differential equations and highlighted advantages compared with difference methods. Brunner et al. [13] used the DG method to solve delay differential equation with one proportional delay. Li et al. [21] applied DG method for delay differential equations with constant delay. Huang et al. [14] improved the global convergence by some accelerate techniques based on the local superconvergence results of DG solutions, and presented the hp-version of the DG method with nonlinear vanishing delays [15]. They also developed the continuous Galerkin (CG) methods for delay differential equations of pantograph type with uniform meshes [16] and quasi-geometric meshes [17,18].

In this paper, we intend to effectively employ the DG method to approximate smooth solutions of the multi-pantograph delay differential equations with uniform meshes. Due to the initial singularity of the forcing term f, solutions of multi-pantograph delay differential equations are singular. We also intend to get the relevant global convergence and local superconvergence of DG solutions for weakly singular multi-pantograph delay differential equations with graded meshes. For simplicity but without loss of generality, we consider the following special multi-pantograph delay differential equation case:

$$u'(t)=a(t)u(t)+b_1(t)u(q_1t)+b_2(t)u(q_2t)+f(t), \quad t\in J:=[0,T], \tag{1.2a}$$

$$u(0)=u_0. \tag{1.2b}$$

We analyze the optimal global convergence and local superconvergence of discontinuous