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Multigrid Method for Poroelasticity Problem by Finite Element Method

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Abstract. In this paper, we will investigate a multigrid algorithm for poroelasticity problem by a new finite element method with homogeneous boundary conditions in two dimensional space. We choose Nédélec edge element for the displacement variable and piecewise continuous polynomials for the pressure variable in the model problem. In constructing multigrid algorithm, a distributive Gauss-Seidel iteration method is applied. Numerical experiments shows that the finite element method achieves optimal convergence order and the multigrid algorithm is almost uniformly convergent to mesh size h and parameter δt on regular meshes.

AMS subject classifications: 65M10, 78A48

Key words: Poroelasticity problem, finite element method, multigrid method.

1 Introduction

General theory describing the consolidation of a porous elastic soil is very important in application, for example, predicting the behavior of foundation resting on a saturated clay is an important problem in foundation engineering. The foundation allows for the occurrence of finite geometry changes and finite elastic strains during the consolidation process. This theory of poroelasticity addresses the time-dependent coupled process between the deformation of porous materials and the fluid flow inside. The governing equations have been cast in a rate form and laws which determine deformation and pore fluid flow are Hookes's law and Darcy's law. The theoretical basis of consolidation was established by Terzaghi [22], then, Biot generalized the theory to three dimensional transient consolidation [4,5]. Since then, poroelastic theory has been used in a diverse range

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of science and engineering application, for instance, CO_2 sequestration in environmental engineering [14] is important applications of poroelasticity. Recently, research in poroelasticity has been a surge in activity, not only because of the application described above, but also due to emerging applications in biomechanics engineering such as biological soft tissue modeling including arterial walls, skin, cardiac muscle and articular cartilage [12, 15, 20, 25].

There are extensive literatures on numerical methods for poroelasticity. The most commonly used numerical discretization are finite element methods for the two fields problem, for example, a continuous Galerkin (CG) element for both displacement and pressure is studied in [13, 16]. Later, finite element method based on three fields (displacement, fluid flux, pore pressure) are analyzed, for instance, couple continuous and discontinuous Galerkin (DG) methods for the displacement with a mixed finite element method for the flow variables are investigated in [17–19]. Also, Yi [27] studied the nonconforming finite element for displacement and standard mixed finite element method for pressure and velocity and Tchonkova [21] used a mixed finite element method which seeks a solution via minimization of a least-squares functionals. Four fields formulation with displacement, stress, pressure and flux is also studied in [26]. Meanwhile, finite difference method on staggered mesh [7], finite volume method [2] for the problem with discontinuous coefficients and weak Galerkin method [10] are also considered for the poroelasticity model problems.

As mesh size becomes much smaller, the scale of discrete equation of poroelasticity model by numerical methods becomes much larger. Therefore, efficient solver for the discrete linear system is important in computing numerical solution. There are lots of studies in constructing fast solvers for poroelascitity model problem, see examples [1,3,6,9,11, 24]. Multigrid algorithm is one of the most efficient iterative scheme which can reduce the computation of discrete equation to $\mathcal{O}(N)$ or $\mathcal{O}(N\log N)$ with N being the scale of the linear system. For the poroelasticity problem, some efficient multigrid algorithm has been designed based on finite difference method. For instance, multigrid algorithm with distributive smoothing on cartesian equidistant grids are constructed in [8,24]. Performance of multigrid methods with two different types of smoother, the decouple smoother (distributive type) and couple smoother (Vanka type) are compared for poroelasticity model in [9] on staggered mesh. In this work, we will study the multigrid method with decoupled smoother by finite element method on triangular mesh.

We will investigate the finite element method as well as the multigrid method for poroelasticity model problem in this study. There are mainly two unknowns, displacement variable $\boldsymbol{u}$ and pore pressure p in the poroelasticity model problems. We constructed the discretization method by introduce an intermediate variable $\omega=-\mathrm{div}\boldsymbol{u}$. For the discrete finite element spaces, we choose the Nédélec edge element for the displacement variable and piecewise continuous Lagrange polynomials for the intermediate variable ω as well as for pore pressure variable. Numerical results show that on regular mesh, this discretization method achieves the optimal order in L^2 norm and corresponding energy norms. We also constructed the multigrid algorithm for the poroelasticity