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An Entropy Stable Scheme for the Multiclass Lighthill-Whitham-Richards Traffic Model

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Abstract. An entropy conservative (EC) numerical flux for the multiclass Lighthill-Whitham-Richards (MCLWR) kinematic traffic model based on the general framework by Tadmor [E. Tadmor, The numerical viscosity of entropy stable schemes for systems of conservation laws, I, Math. Comput., 49 (1987), pp. 91–103] is proposed. The approach exploits the existence of an entropy pair for a particular form of this model. The construction of EC fluxes is of interest since in combination with numerical diffusion terms they allow one to design entropy stable schemes for the MCLWR model. In order to obtain a higher-order accurate scheme and control oscillations near discontinuities, a third-order WENO reconstruction recently proposed by Ray [D. Ray, Third-order entropy stable scheme for the compressible Euler equations, in C. Klingenberg and M. Westdickenberg (eds.), Springer Proc. Math. Stat., 237, pp. 503–515] is used. Numerical experiments for different classes of drivers are presented to test the performance of the entropy stable scheme constructed with the entropy conservative flux proposed.

AMS subject classifications: 35L65, 35L45, 765M06, 6T99, 90B20
Key words: Multiclass Lighthill-Whitham-Richards traffic model, system of conservation laws, entropy conservative flux, entropy stable scheme.

1 Introduction

The aim of this paper is to introduce an entropy conservative flux for the multiclass Lighthill-Whitham-Richards kinematic traffic model (MCLWR). This model is a generalization of the well-known Lighthill-Whitham-Richards model [17,21] to multiple classes

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of drivers and were independently formulated by Wong and Wong [29] and Benzoni-Gavage and Colombo [1]. The model is described by the nonlinear and spatially one-dimensional systems of conservation laws

$$\partial_t \boldsymbol{\rho} + \partial_x \boldsymbol{f}(\boldsymbol{\rho}) = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \tag{1.1}$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}(x,t) = (\rho_1,\cdots,\rho_N)^{\mathrm{T}}$ is the vector of densities, that is, for each $i=1,\cdots,N$, ρ_i is the density of vehicles belonging to the class or species i, and $\boldsymbol{f}(\boldsymbol{\rho}) = (f_1(\boldsymbol{\rho}),\cdots,f_N(\boldsymbol{\rho}))^{\mathrm{T}}$ is the flux vector. Under the assumptions that drivers of each class adjust their velocity to the total traffic density $\rho = \rho_1 + \cdots + \rho_N$, and all drivers adjust their velocity in the same way, the MCLWR model is defined by the relationship

$$f_i(\boldsymbol{\rho}) = v_i^{\max} \rho_i \phi(\rho), \quad i=1,\cdots,N, \tag{1.2}$$

where $v_i^{\max}$ is the maximum velocity attained by cars in class i (free flowing speed) and $\phi(\rho)$ is a function describing the behavior of drivers. Some standard expressions for ϕ include the Greenshields model [13]

$$\phi(\rho) = 1 - \rho/\rho_{\max}, \tag{1.3}$$

where $\rho_{\max}$ is a maximum traffic density corresponding to the "bumper-to-bumper" situation, or the Drake model [6]

$$\phi(\rho) = \exp\left(-(\rho/\rho_0)^2/2\right). \tag{1.4}$$

It is further assumed that $0 < v_1^{\max} < \cdots < v_N^{\max}$. Broad introductions to mathematical models for vehicular traffic, in particular on the choice of velocity functions within and the extensions of the LWR model, are provided in [11,27,28].

It is well known that the system (1.1), (1.2) is strictly hyperbolic in the interior of the phase space for (1.1),

$$\mathcal{D} := \{\boldsymbol{\rho} = (\rho_1,\cdots,\rho_N)^{\mathrm{T}} \in \mathbb{R}^N : \rho_1 \geq 0,\cdots,\rho_N \geq 0,\ \rho = \rho_1 + \cdots + \rho_N \leq \rho_{\max}\},$$

and admits a separable entropy function (see below for detailed explanations) for arbitrary numbers N of driver classes, that is, of scalar equations in (1.1). The latter property is exceptional for systems of conservation laws of practical interest but does hold for the MCLWR models. On the other hand, it is a pre-requisite for the applicability of entropy stable schemes for systems of conservation laws that were proposed in a series of papers including [7–9, 24–26]. It is the purpose of this paper to demonstrate that entropy stable schemes, based on the use of an entropy conservative numerical flux, in combination with weighted essentially non-oscillatory (WENO) reconstructions indeed provide an accurate method for the numerical solution of the MCLWR model.

The hyperbolicity of the system (1.1), (1.2) has been studied by many authors. Benzoni-Gavage and Colombo [1] proved the hyperbolicity of the model by showing