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An H(div)-Conforming Finite Element Method for the Biot Consolidation Model

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Abstract. An $H(\mathrm{div})$-conforming finite element method for the Biot's consolidation model is developed, with displacements and fluid velocity approximated by elements from BDM_k space. The use of $H(\mathrm{div})$-conforming elements for flow variables ensures the local mass conservation. In the $H(\mathrm{div})$-conforming approximation of displacement, the tangential components are discretised in the interior penalty discontinuous Galerkin framework, and the normal components across the element interfaces are continuous. Having introduced a spatial discretisation, we develop a semi-discrete scheme and a fully discrete scheme, prove their unique solvability and establish optimal error estimates for each variable.

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1. Introduction

Poroelasticity [3] attracts more and more attention because of its important role in various applications, including carbon sequestration in environment engineering, seismic wave propagation in earthquake prediction, surface subsidence, evolution of fractured reservoirs during gas production, and biomechanical descriptions of tissues and bones. The models describe the interaction of fluid flows and deformable elastic porous media saturated in the fluid. Here, we deal with the Biot consolidation model, with the motion of fluid in porous media described by the Darcy's law and deformations governed by the linear elasticity.

The complexity of the Biot model and geometrical properties of the domain often prevent from finding analytical solutions of the problem so that numerical simulations got very

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popular — cf. Refs. [8, 10, 16, 24–31, 35, 36]. Since both fluid dynamics and elasticity are involved, it is important to have effective methods, which could approximate the relevant physical processes. Unfortunately, various complications in elasticity and fluid mechanics are often translated into the model approximations — e.g. continuous Galerkin approximations of the displacements may cause locking or nonphysical pressure oscillation [5,26,29] in the linear elasticity part. In order to eliminate the locking phenomenon, one can use a mixed finite element method [22, 35], nonconforming finite elements [34] and discontinuous [29] or weak Galerkin [10, 18, 31] methods. On the other hand, in incompressible fluid flow models, standard Stokes elements such as Taylor-Hood and Mini elements, do not satisfy the divergence constraints strongly or globally and therefore are not mass conservative [12, 13, 19].

In this work, we follow the strategy in [12, 13, 32] and adopt $H(\text{div})$-conforming finite elements for displacements with the aim to relax the H^1-conformity of displacements. The advantage of such a discretisation is two-fold: on one hand, the normal components of displacements across elements are continuous and therefore are locally conservative and on the other hand the tangential components are discretised via an interior penalty discontinuous Galerkin method. This allows us to overcome the locking phenomenon and the pressure oscillation [19, 30, 36]. Note that the use of $H(\text{div})$-conforming finite elements in discontinuous Galerkin (DG) method framework is proposed in [12, 32] and was applied to the Navier-Stokes equations of fluid flow in [13]. Later on, the method has been extended to the Darcy-Stokes interface problems [11, 20], to the Brinkman problem [21] and to a magnetic induction model [9]. In the fluid part of the Biot model, the governing equation occurs from the Darcy's law, and if the mixed form of the Darcy's law is used, it is natural to employ an $H(\text{div})$-conforming finite element discretisation of the flow variables, since it guarantees mass conservation. Here, we adopt the Brezzi-Douglas-Marini (BDM_k) space for both displacements and flow variables. Moreover, the finite element method here provides a unified approach to flow variables and displacements. This work can be regarded as a further development of $H(\text{div})$-conforming finite element methods for Biot's problems. Using the approach in [27,28,34,36], we present a detailed analysis of the method. In particular, for both semi-discrete and fully discrete schemes for the Biot model, we show the existence and uniqueness of approximate solutions and derive an optimal convergence rate for each variable.

The rest of this paper is organised as follows. In Section 2, the Biot consolidation model, functional spaces and corresponding weak formulation are introduced. A spatial semi-discrete scheme involving $H(\text{div})$-conforming elements is considered in Section 3. The existence and uniqueness results are proved and a priori error estimates for the semi-discrete scheme are derived. Section 4 is devoted to a fully discrete numerical scheme based on the backward Euler time discretisation. Our conclusions are in Section 5.

2. Biot's Consolidation Model and Its Weak Formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with a Lipschitz boundary $\partial\Omega$ and $(0, T]$ a time interval. We consider the following Biot's consolidation model: