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Fast Algorithms for Boundary Integral Equations on Elliptic Domains and Related Inverse Problems

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Abstract. Fast algorithms for boundary integral equations connected with Robin boundary value problem for the Laplace equation in domains with ellipse or close to ellipse boundaries are developed. It is shown that the coefficient matrices of discretisation systems have a special structure. This fact is used to develop a fast algorithm for matrix vector multiplication and to implement it in the numerical methods used. Such an approach is especially helpful in numerical methods for inverse problems, since many methods of their solution repeatedly use forward solvers. The efficiency of the methods is illustrated by numerical examples.

AMS subject classifications: 65F22, 65R32

Key words: Robin boundary value problem, Robin inverse problem, circulant-like matrix, fast iterative methods.

1. Introduction

We consider a Robin boundary value problem for the Laplace equation in a smooth connected open domain $\Omega \subset \mathbb{R}^2$ with the boundary Γ — viz.

$$\begin{aligned} &\Delta u = 0 \quad \text{in} \quad \Omega, \\ &\frac{\partial u}{\partial \nu} + pu = g \quad \text{on} \quad \partial\Omega = \Gamma, \end{aligned} \tag{1.1}$$

where ν is the outward unit normal direction of the boundary and p the Robin coefficient, nonnegative and not identically equal to zero on Γ.

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The Robin boundary value problem (1.1) serves as a mathematical model for various physical phenomena and is widely used in applications, including heat distribution of thermal conducting materials and electrical potentials in semiconductors with contacts through the boundary. For known Γ, p and g, the solution u of (1.1) is uniquely determined. More interesting problems often arising in applications, consist in recovering the term p or part of Γ in (1.1) from additional data — e.g. from additional measurements of the solution u on an accessible part of the boundary Γ. This extra information on the solution is to be used to extract various physical parameters of interest, such as the Robin coefficient p that describes the quality of metal-to-silicon contact on the boundary in a transistor, or an inaccessible part of Γ that represents a desirable material profile. Such inverse problems have been extensively studied in recent years both by analytical and numerical methods — cf. [1, 3–6, 8, 13, 15] and references therein.

One of popular approaches to the problem (1.1) consists in transforming the boundary value problem for the Laplace equation into an equivalent boundary integral equation (BIE) for u on Γ and applying suitable numerical methods to the resulting integral equation. This approach has been successfully used in solving inverse problems for (1.1) and we follow it here. Transforming (1.1) into a boundary integral equation on an ellipse and discretising it, we exploit the structure of the coefficient matrix of the resulting discrete system and construct a fast algorithm for the matrix-vector multiplication, which requires only $\mathcal{O}(n\log n)$ operations instead of standard $\mathcal{O}(n^2)$. This is also an essential ingredient in iterative methods for the forward problem. We adopt the idea to construct numerical methods, which work faster than conventional algorithms for boundaries Γ close to an ellipse specified in Section 2.3. The solution of inverse problems often depends on repeated use of forward solvers, so that a faster forward solver can essentially speed up the corresponding numerical methods. We illustrate the efficiency of these fast algorithms by solving two inverse problems — viz. the recovery of the Robin coefficient p on an ellipse and the recovery of a part of non-elliptic segment on Γ. In both cases, measurements of u on another part of Γ are used. Numerical examples show a reduced computation time in comparison with conventional algorithms that do not exploit special structure of arising matrices. This property is especially valuable if the size of discrete system grows.

The outline of the paper is as follows. In Section 2, the Robin problem (1.1) is reduced to an equivalent boundary integral equation. Assuming that Γ is an ellipse, we then construct a fast algorithm for the discretisation of the linear system. This algorithm is also adopted in the case where a small part of Γ differs from ellipse arc. In Section 3, we apply fast algorithms to two inverse problems: the recovery of the Robin coefficient and the recovery of a part of the boundary from the measurements of u on another part of Γ. Numerical results presented in Section 4, illustrate the efficiency of the methods. Our concluding remarks are in Section 5.

2. Boundary Integral Equations and Fast Methods

Assume that the domain Ω has a C^2-boundary Γ, and let $p(x) \geq 0$ on Γ with $\operatorname{supp}(p) \neq \emptyset$. By Green's formula, the solution u of (1.1) in Ω can be represented via its boundary values