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Vector Solutions with Prescribed Component-Wise Nodes for a Schrödinger System

Zhaoli Liu[1,*] and Zhi-Qiang Wang[2,3]

[1] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[2] *Center for Applied Mathematics, Tianjin University, Tianjin 300072, China*
[3] *Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322, USA*

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Abstract. For the Schrödinger system

$$\begin{cases} -\Delta u_j+\lambda_j u_j=\sum_{i=1}^{k}\beta_{ij}u_i^2u_j \quad \text{in } \mathbb{R}^N, \\ u_j(x)\to 0 \quad \text{as } |x|\to\infty, \quad j=1,\cdots,k, \end{cases}$$

where $k\geq 2$ and $N=2,3$, we prove that for any $\lambda_j>0$ and $\beta_{jj}>0$ and any positive integers p_j, $j=1,2,\cdots,k$, there exists $b>0$ such that if $\beta_{ij}=\beta_{ji}\leq b$ for all $i\neq j$ then there exists a radial solution $(u_1,u_2,\cdots,u_k)$ with u_j having exactly p_j-1 zeroes. Moreover, there exists a positive constant C_0 such that if $\beta_{ij}=\beta_{ji}\leq b$ $(i\neq j)$ then any solution obtained satisfies

$$\sum_{i,j=1}^{k}|\beta_{ij}|\int_{\mathbb{R}^N}u_i^2u_j^2\leq C_0.$$

Therefore, the solutions exhibit a trend of phase separations as $\beta_{ij}\to-\infty$ for $i\neq j$.

Key Words: Vector solution, prescribed component-wise nodes, Schrödinger system, variational methods.

AMS Subject Classifications: 35A15, 35J10, 35J50

1 Introduction

We consider the coupled Schrödinger system

$$\begin{cases} -\Delta u_j+\lambda_j u_j=\sum_{i=1}^{k}\beta_{ij}u_i^2u_j \quad \text{in } \mathbb{R}^N, \\ u_j(x)\to 0 \quad \text{as } |x|\to\infty, \quad j=1,\dots,k, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `zliu@cnu.edu.cn` (Z. Liu), `zhi-qiang.wang@usu.edu` (Z.-Q. Wang)

where $k\geq 2$ and $N=2,3$. We assume $\lambda_j>0$, $\beta_{jj}>0$, and $\beta_{ij}=\beta_{ji}$ $(j\neq i)$ are constants.

This type of systems arises when one considers standing wave solutions of time-dependent k-coupled Schrödinger systems of the form

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_j=\Delta\Phi_j-V_j(x)\Phi_j+\mu_j|\Phi_j|^2\Phi_j+\Phi_j\sum\limits_{i=1,i\neq j}^{k}\beta_{ij}|\Phi_i|^2 & \text{in } \mathbb{R}^N,\\ \Phi_j=\Phi_j(x,t)\in\mathbb{C} & t>0,\quad j=1,\cdots,k. \end{cases} \tag{1.2}$$

These systems of equations, also known as coupled Gross-Pitaevskii equations, have applications in many physical problems (see [1, 27]) in particular in Bose-Einstein condensates theory for multispecies Bose-Einstein condensates (see [10, 14, 32, 40]) which have been studied intensively in the last twenty years. Physically, β_{jj} and β_{ij} $(i\neq j)$ are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length determines whether the interactions of states are repulsive or attractive. In the attractive case ($\beta_{ij}>0$ for $i\neq j$) the components of a vector solution tend to go along with each other leading to synchronization. And in the repulsive case ($\beta_{ij}<0$ for $i\neq j$) the components tend to segregate component-wisely, leading to phase separations and much more complicated behaviors of solutions.

Mathematical properties of systems of nonlinear Schrödinger equations have been studied extensively in recent years; see, e.g., [2–6,8,10–13,15–23,25,26,28,30,33–39,41–43] and references therein. Phase separation has been proved in several cases with constant potentials such as in the work [4, 10–12, 30, 38, 42, 43] as the coupling constant β tends to negative infinity in the repulsive case. It is quite natural to assert that due to segregation in the repulsive case the structures of vector solutions are much richer and more complex. In particular, in the repulsive case, multiplicity of positive solutions has been established in [12, 38, 39, 42], multiple non-trivial vector solutions were constructed in [25, 26], and multiple sign-changing solutions have been given in [22, 23, 35]. There has been progress for the mixed coupling cases and, due to the repulsive effects, there exist many distinct types of solutions exhibiting partial synchronization and partial segregation phenomena (see, e.g., [8, 31, 33, 34, 37]). Due to the above existing work, we remark that there are new difficulties in dealing with the existence of multiple sign-changing solutions. First, there are many semi-trivial solutions due to systems collapsing, i.e., there are solutions of the form in which one or more components are zero so they are solutions of systems of fewer number of equations. Second, there can exist (infinitely) many positive solutions. For the totally symmetric case ($\lambda_j=\lambda>0$ and $\mu_j=\mu>0$ for all j, and $\beta_{ij}=\beta$ for all $i\neq j$), in [38] radial solutions with domain separations are constructed using variational methods and perturbation methods for k-systems, and in [12, 43] minimax method is used to give infinitely many radial positive solutions for 2-systems (see also [39] for generalizations to the k-systems). These radial solutions demonstrate segregation nature. Segregated radial solutions were obtained in repulsive case in [4] by global bifurcation methods for systems (1.1) with $k=2$ establishing the existence of infinitely many branches of radial solutions with the property that a weighted difference between the two components of solutions along the m-th branch has exactly m nodal domains. While these results are all