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On Conformal Metrics with Constant Q-Curvature

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Abstract. We review some recent results in the literature concerning existence of conformal metrics with constant Q-curvature. The problem is rather similar to the classical Yamabe problem: however it is characterized by a fourth-order operator that might lack in general a maximum principle. For several years existence of geometrically admissible solutions was known only in particular cases. Recently, there has been instead progress in this direction for some general classes of conformal metrics.

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AMS Subject Classifications: 35B33, 35J35, 53A30, 53C21

1 Introduction

A classical problem in conformal geometry is the *Yamabe problem*, consisting in deforming a background metric on a compact manifold (M,g) so that its scalar curvature becomes constant. This can be considered as an extension of the classical uniformization problem for two-dimensional surfaces and has received a lot of attention in the literature, see [34] for a general introduction to the problem.

The scalar curvature of a manifold transforms conformally according to the law

$$L_g u + R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}}, \quad \tilde{g} = u^{\frac{4}{n-2}} g, \tag{1.1}$$

where L_g is the *conformal laplacian*, defined by

$$L_g \phi = -\frac{4(n-1)}{(n-2)} \Delta_g \phi + R_g \phi.$$

The latter operator transforms covariantly, namely one has

$$L_g(u\phi) = u^{\frac{n+2}{n-2}} L_{\tilde{g}}(\phi). \tag{1.2}$$

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The Yamabe problem then amounts to finding a positive solution to (1.1) with $R_{\tilde{g}}$ equal to a constant. This constant can be viewed as a Lagrange multiplier when considering the following minimization problem

$$Y(g)=\inf_{u\in W^{1,2}(M,g),u\neq 0}\frac{\int_M uL_g u d\mu_g}{\left(\int_M |u|^{\frac{2n}{n-2}}d\mu_g\right)^{\frac{n-2}{n}}}. \tag{1.3}$$

It can be proved using (1.1) and (1.2) that the latter quantity is conformally invariant.

Since the Sobolev embedding $W^{1,2}(M,g)\hookrightarrow L^{\frac{2n}{n-2}}(M,g)$ is not compact, it is a-priori not clear whether a minimizer exists. In [44] the problem was attacked by lowering the exponent by a small amount and trying to pass to the limit: however the original proof of convergence was faulty. In [41] existence of minimizers was shown provided $Y(g)$ is smaller than a given positive dimensional constant (and in particular when it is negative or zero). In [1] it was shown via a compactness argument that minimizers exist provided $Y(g)<Y(g_{S^n})$, which was verified in dimension $n\geq 6$ if (M,g) is not conformally equivalent to the standard sphere. Under this latter assumption, in [39] the same strict inequality was proved in the complementary cases, i.e., for $n=3,4,5$ or (M,g) locally conformally flat. While the argument in [1] exploited a local geometric expansion involving the Weyl tensor, the one in [39] relied on the *Positive Mass Theorem* in general relativity.

We next discuss some higher-order analogue of the above problem. In [2] T.Branson introduced the following fourth-order operator in dimension $n\geq 5$:

$$P_g u=\Delta_g^2 u-div_g\left(a_n R_g g+b_n Ric_g\right)du+\frac{n-4}{2}Q_g u,$$

where

$$a_n=\frac{(n-2)^2+4}{2(n-1)(n-2)},\quad b_n=-\frac{4}{n-2}.$$

The function

$$Q_g=\frac{1}{2(n-1)}\Delta_g R_g+\frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2-\frac{2}{(n-2)^2}\left|Ric_g\right|^2$$

is the so-called *Q-curvature* (see [14,18,29] for more general operators and formulas). As for L_g, the operator P_g is conformally covariant: if $\tilde{g}=u^{\frac{4}{n-4}}g$ is a conformal metric to g, then for all $\phi\in C^\infty(M)$ we have

$$P_g(u\phi)=u^{\frac{n+4}{n-4}}P_{\tilde{g}}(\phi). \tag{1.4}$$

Moreover one has the following conformal transformation law

$$P_g u=Q_{\tilde{g}}u^{\frac{n+4}{n-4}},\quad \tilde{g}=u^{\frac{4}{n-4}}g, \tag{1.5}$$

analogous to (1.1).