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Convergence and Quasi-Optimality of an Adaptive Continuous Interior Penalty Finite Element Method

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Abstract. An adaptive continuous interior penalty finite element method (ACIPFEM) for symmetric second order linear elliptic equations is considered. Convergence and quasi-optimality of the ACIPFEM are proved. Compared with the analyses for the adaptive finite element method or the adaptive interior penalty discontinuous Galerkin method, extra works are done to overcome the difficulties caused by the additional penalty term. Numerical tests are provided to verify the theoretical results and show advantages of the ACIPFEM.

AMS subject classifications: 65N12, 65N15, 65N30, 78A40

Key words: Continuous interior penalty finite element method, adaptive algorithm, convergence, quasi-optimality.

1 Introduction

Let Ω be a bounded, polyhedral domain in $\mathbb{R}^d$, $d=2,3$. We analyze an adaptive continuous interior penalty finite element method (ACIPFEM) for the second order elliptic partial differential equation

$$L(u):=-\mathrm{div}(A\nabla u)+cu=f \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega. \tag{1.1}$$

Precise conditions on given data (A,c) and f are specified later. The choice of boundary condition is made for ease of presentation, since similar results are valid for other boundary conditions.

The continuous interior penalty finite element method (CIPFEM), which was first proposed by Douglas and Dupont [28] for elliptic and parabolic problems in 1970's and then successfully applied to convection-dominated problems as a stabilization technique [12–

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16], uses the same approximation space as the finite element method (FEM) but modifies the bilinear form of the FEM by adding a least squares term penalizing the jump of the gradient of the discrete solution at mesh interfaces. More recently, the CIPFEM has shown great potential for simulating Helmholtz scattering problems with high wave number [29,44,49]. We remark that the idea of penalizing the jump of the gradient of the discrete solution is also used in the interior penalty Galerkin methods (IPDG) [4,11,30–32,48].

The adaptivity has been a fundamental technique for about four decades in FEM, IPDG methods, and many other methods to deal with various singularities. Although there have been many works on the convergence analysis of the adaptive finite element methods (AFEM) or the adaptive interior penalty discontinuous Galerkin methods (AIPDG) [2,3,5,8,9,18–22,26,27,34,35,38,39,41,45,46], not much work has been done for the ACIPFEM. We refer to [16] for an a posteriori error estimate for the CIPFEM.

The purpose of this paper is to prove convergence and quasi-optimality for the ACIPFEM based on an a posteriori error estimator of residual type and the Dörfler's marking strategy. The basic idea of the analysis is to mimic that of the AFEM (cf. [20]), but some essential difficulties caused by the introduced penalty term need to be treated specially. To be precise, the variational formulation of the CIPFEM is not consistent with the continuous problem for weak solution u just in $H^1(\Omega)$ and hence the Galerkin orthogonality is no longer valid. We surmount this difficulty by using an approximate Galerkin orthogonality in which the penalty term is regarded as a perturbation, and estimating the perturbation very carefully (see, e.g., Theorem 2.1, Lemma 3.3, Lemma 4.1). Finally, under the same conditions as those of AFEM, we may prove the convergence and quasi-optimality of the ACIPFEM with the penalty parameter γ satisfies $0\leq\gamma\lesssim 1$.

The rest of this paper is organized as follows. In Section 2 we introduce the CIPFEM and derive its upper and lower error estimates. Section 3 is devoted to state the adaptive algorithm and prove its contraction property. The quasi-optimality of the ACIPFEM is proved in Section 4. In Section 5, we apply the ACIPFEM to the crack problem to verify the theoretical findings and to a Helmholtz problem with large wave number on a L-shaped domain to show advantages of the ACIPFEM.

Throughout the paper, we use the shorthand notation $A\lesssim B$ whenever $A\leq CB$ with a constant C independent of parameters which A and B may depend on. $A\eqsim B$ is a shorthand notation for the statement $A\lesssim B$ and $B\lesssim A$.

2 The CIPFEM and its a posteriori error estimates

In this section, we first state the assumptions on given data and the formulation of CIPFEM, and then deduce the a posteriori error estimates including upper and lower bounds. Throughout this paper, the standard space, norm and inner product notations are adopted. Their definitions can be found in [10, 23]. In particular, $(\cdot,\cdot)_\omega$, $\langle\cdot,\cdot\rangle_\omega$ and $\|\cdot\|_\omega$ denote the L^2-inner products and L^2-norm on $L^2(\omega)$ space, respectively. Denote by