Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **11**, No. 2, pp. 360-380

DOI: 10.4208/aamm.OA-2018-0115
April 2019

A Class of Weak Galerkin Finite Element Methods for the Incompressible Fluid Model

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Received 25 April 2018; Accepted (in revised version) 25 September 2018

Abstract. In this paper, we propose a weak Galerkin finite element method (WG) for solving the stationary incompressible Stokes equation in two or three dimensional space. The weak Galerkin finite element method is highly flexible by allowing the use of discontinuous functions on arbitrary polygons or polyhedra with certain shape regularity. However, since additional variables are introduced, the computational cost is much higher. Our new method can significantly reduce the computational cost while maintaining the accuracy. Optimal error orders are established for the weak Galerkin finite element approximations in various norms. Some numerical results are presented to demonstrate the efficiency of the method.

AMS subject classifications: 65N15, 65N30, 76D07

Key words: Incompressible Stokes equation, weak Galerkin finite element method, discrete weak gradient, Schur complement.

1 Introduction

Flows occur everywhere. Without flows we would not even exist. Thus the study of flows is always one of the hot topics. From 1687, the Isac Newton do a famous experiment of viscous flow. He had found that the resistance and velocity gradient of almost all common fluids have linear relationships. It makes people reasonably know the viscous flow. Then, the Euler equations had been proposed in 1755. After the many investigators of efforts, the ideal fluid without the viscidity have obtained gradually the perfect level. Nevertheless, there was a wide gap between the solution of the ideal fluid and the test result of the real fluid, sometimes even be on the contrary. Until 1821, Naiver and

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other specialists had started to consider the intermolecular forces in the Euler equations. Then, the George Gabriel Stokes had made the viscosity coefficient μ represent the intermolecular forces and completed the Naiver-Stokes equations. Finally, the fundamental equations of the viscous fluid mechanics had been established. When the viscous force and the ratio of the advection inertial force of a flow is very small, it is called a Stokes flow and the equations describing the flow is called Stokes equations. In this paper, we consider the following Stokes equations for unknown velocity function $\mathbf{u}$ and the pressure function p in the domain Ω

$$-\mu\Delta\mathbf{u}+\nabla p=\mathbf{f} \qquad \text{in } \Omega, \tag{1.1a}$$
$$\nabla\cdot\mathbf{u}=0 \qquad \text{in } \Omega, \tag{1.1b}$$
$$\mathbf{u}=\mathbf{g} \qquad \text{on } \partial\Omega, \tag{1.1c}$$

where the Ω is a polygonal or polyhedra domain in $\mathbb{R}^d\,(d=2,3)$. The function p satisfies

$$\int_\Omega p\, ds=0.$$

Here $\mu>0$ is the viscosity coefficient, $\mathbf{f}$ is the source term, and $\mathbf{g}$ satisfies the compatibility condition

$$\int_\Omega \mathbf{g}\cdot\mathbf{n}\, ds=0,$$

where $\mathbf{n}$ is the unit outward normal vector on the domain boundary $\partial\Omega$. For simplicity, we assume that $\mu=1$ and consider the homogeneous Drichlet boundary condition $\mathbf{g}=\mathbf{0}$.

The weak forms for the Stokes problems (1.1a)-(1.1c) in the primal velocity-pressure formulation are to find $(\mathbf{u};p)\in[H_0^1(\Omega)]^d\times L_0^2(\Omega)$ such that

$$(\nabla\mathbf{u},\nabla\mathbf{v})-(p,\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v}), \tag{1.2a}$$
$$(q,\nabla\cdot\mathbf{u})=0, \tag{1.2b}$$

for all $(\mathbf{v};q)\in[H_0^1(\Omega)]^d\times L_0^2(\Omega)$. This is a typical saddle point problem, so the stability and uniqueness of solution can be proved by inf-sup condition. Before the last several decades, the finite element method has been solved the Stokes problems (1.1a)-(1.1c) on the weak forms (1.2a)-(1.2b). The detailed analysis can be found in [17,19].

If $(\mathbf{u};p)\in[H_0^1(\Omega)]^d\times(L_0^2(\Omega)\cap H^1(\Omega))$, testing the Eq. (1.1b) and integration by parts can be obtained the gradient-gradient weak form

$$(\nabla\mathbf{u},\nabla\mathbf{v})+(\nabla p,\mathbf{v})=(\mathbf{f},\mathbf{v}), \tag{1.3a}$$
$$(\nabla q,\mathbf{u})=0, \tag{1.3b}$$

for all $(\mathbf{v};q)\in[H_0^1(\Omega)]^d\times(L_0^2(\Omega)\cap H^1(\Omega))$. The parameters of the Theorem 5.4 in the paper [3] are taken by $m=0$ and $r=1$, so we get L^2 regularity. When the pressure function $p\in H^1(\Omega)$, the gradient function ∇p exists. In this case, the variational forms (1.2a)-(1.2b)