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Lump and Interaction Solutions of Linear PDEs in $(3+1)$-Dimensions

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Abstract. Linear partial differential equations in $(3+1)$-dimensions consisting of all mixed second-order derivatives are considered, and Maple symbolic computations are made to construct their lump and interaction solutions, including lump-periodic, lump-kink and lump-soliton solutions.

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1. Introduction

Lump solutions are special exact solutions of partial differential equations (PDFs),which describe important wave phenomena [1,29]. Specific lumps can be obtained from solitons through taking long wave limits [30]. Other classes of solutions to integrable equations include positons and complxitons [16,35], and interaction solutions [26], which exhibit more diverse nonlinear wave phenomena.

From a mathematical point of view, soliton solutions are exponentially localised in time and in all space directions, whereas lump solutions are rationally localised in all space

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directions. Let P be a polynomial, and D_x and D_t be the Hirota bilinear derivatives. Based on the Hirota bilinear form

$$P(D_x, D_t)f \cdot f = 0,$$

the corresponding N-soliton solution in $(1+1)$-dimensions can take the form

$$f = \sum_{i,j=1}^{N} \exp(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}),$$

where $\mu_j \in \{0,1\}$, $j = 1, 2, \cdots, N$, and

$$\xi_i = k_i x - \omega_i t + \xi_{i,0}, \quad 1 \le i \le N,$$
$$\mathrm{e}^{a_{ij}} = -\frac{P(k_i - k_j, \omega_j - \omega_i)}{P(k_i + k_j, \omega_j + \omega_i)}, \quad 1 \le i < j \le N,$$

with the wave numbers k_i and the wave frequencies ω_i satisfying the dispersion relation, and $\xi_{i,0}$ being arbitrary shifts.

It is known [21] that the KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0$$

has the lump solution

$$u = 2(\ln f)_{xx}, \quad f = \big(a_1 x + a_2 y + a_3 t + a_4\big)^2 + \big(a_5 x + a_6 y + a_7 t + a_8\big)^2 + a_9,$$

where

$$a_3 = \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2, a_2 a_5 a_6}{{a_1}^2 + {a_5}^2}, \quad a_7 = \frac{2a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2}, \quad a_9 = \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1 a_6 - a_2 a_5)^2},$$

and $a_1 a_6 - a_2 a_5 \neq 0$. The last condition guarantees the rational localisation in all directions in the (x, y)-plane. There are many other integrable equations with lump solutions — e.g. three-dimensional three-wave resonant interaction [8], BKP equation [5, 38], Davey-Stewartson equation II [30], Ishimori-I equation [7] — cf. also Refs. [27, 46]. Moreover, non-integrable equations can also have lump solutions [2, 24, 43, 44], and there are interaction solutions of nonlinear integrable equation in (2+1)-dimensions, including lump-soliton interaction solutions [25, 39, 41, 42] and lump-kink interaction solutions [9, 31, 45, 48]. In $(3+1)$-dimensions, only the integrable Jimbo-Miwa equation has been known to have lump-type solutions, rationally localised in almost all (but not all) space directions. On the other hand, all analytical rational solutions of the $(3+1)$-dimensional Jimbo-Miwa equation in [22, 40, 47] and of the $(3+1)$-dimensional Jimbo-Miwa like equation in [6] are not rationally localised in all space directions, either. Therefore, in $(3+1)$-dimensions, lump and interaction solutions of PDEs are interesting objects to study.

The aims of this work is to show the existence of lump and interaction solutions of PDEs in $(3+1)$-dimensions. A class of particular examples of equations in $(3+1)$-dimensions is