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A Preconditioned Fast Finite Volume Method for Distributed-Order Diffusion Equation and Applications

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Abstract. A Crank-Nicolson finite volume scheme for the modeling of the Riesz space distributed-order diffusion equation is proposed. The corresponding linear system has a symmetric positive definite Toeplitz matrix. It can be efficiently stored in $\mathcal{O}(NK)$ memory. Moreover, for the finite volume scheme, a fast version of conjugate gradient (FCG) method is developed. Compared with the Gaussian elimination method, the computational complexity is reduced from $\mathcal{O}(MN^3+NK)$ to $\mathcal{O}(l_A MN\log N+NK)$, where l_A is the average number of iterations at a time level. Further reduction of the computational cost is achieved due to use of a circulant preconditioner. The preconditioned fast finite volume method is combined with the Levenberg-Marquardt method to identify the free parameters of a distribution function. Numerical experiments show the efficiency of the method.

AMS subject classifications: 35R11, 65F08, 65F10, 65M08, 65T50

Key words: Distributed-order diffusion equation, finite volume method, fast conjugate gradient method, circulant preconditioner, parameter identification.

1. Introduction

In the past few decades, fractional partial differential equations (PDEs) have been widely used to model complex physical phenomena with long-range time memory and spatial interactions [3, 7, 27, 29, 36]. Systematic introduction to fractional calculus and fractional differential equations can be found in Refs. [33, 35].

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Unlike the PDEs of integer-order, analytical solutions of fractional PDEs are rarely available, so that numerical methods have to be employed — cf. Refs. [8,9,13,17,18,24–26,30, 31,39,44,46]. However, the nonlocal nature of fractional differential operators leads to dense stiffness matrices and/or long tails in the time direction. Thus, traditional approximation methods using numerical discretisation, have a high computational cost, especially in multidimensional situations. In 2010, Wang *et al.* [41] proposed a direct fast finite difference method for space-fractional diffusion equations, which retained the same accuracy as regular finite difference methods but required only $\mathscr{O}(N)$ memory storage with the computational cost $\mathscr{O}(N\log^2 N)$. After that, fast solution methods have been extended to various fractional PDEs, including space-fractional PDEs [19,21,34,42,43], time-fractional PDEs [20,45] and space-time-fractional PDEs [10,11,15].

Recently, Li *et al.* [23] considered a finite volume method for a distributed-order space-fractional model and proved the unconditional stability, convergence and the second order accuracy of the method, both in space and time. Here, we want to develop a preconditioned fast finite volume method for a distributed-order space-fractional model and apply it to an inverse problem to determine the free parameters of the corresponding distribution function. Starting with the investigation of the matrix structure of the method and its efficient storage, we then develop a preconditioned fast conjugate gradient (PFCG) method based on a circulant preconditioner and fast matrix-vector multiplication. Numerical experiments show a largely reduced CPU usage, hence the method is well suited to large-scale modeling and simulation. Let us recall that various application problems require the identification of free parameters in the corresponding mathematical models — e.g. given experimental data, determine a parameter by minimising the difference between the numerical output and experimental data. Such procedures are usually considered as inverse problems [6,12], and in this work we develop a PFCG-based optimisation algorithm, which is based on the Levenberg-Marquardt iterative method with the Armijo rule. It is numerically tested, including the situations when the observation data contaminated by random noise. The numerical tests show the efficiency and accuracy of the method proposed.

The rest of the paper is organised as follows. In Section 2, we consider the Riesz space distributed-order diffusion equation and describe the corresponding finite volume approximations. Section 3 discusses the structure of the finite volume scheme matrix and its efficient storage. In Section 4, we develop a PFCG iterative method for the finite volume scheme and test its efficiency. Section 5 is devoted to the identification of free parameters for a distributed-order diffusion equation with the distribution function (2.2). We note that numerical experiments show the strong performance of the method. Our concluding remarks are in Section 6.

2. A Diffusion Equation and Finite Volume Approximations

In this paper, we develop a preconditioned fast finite volume method for the Riesz space distributed-order diffusion equation