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A Numerical Method for the Time Fractional Model of Generalised Second Grade Fluid Through Porous Media

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Abstract. A numerical method for the generalised second grade fluid through porous media with anomalous diffusion is considered. The method is based on a combination of finite differences in time and a spectral method in space directions. The convergence of the method is rigorously proved and theoretical error estimates are established. The numerical scheme is unconditionally stable and provides a high accuracy if the solution is smooth enough. The results of numerical simulations are consistent with theoretical findings.

AMS subject classifications: 76A10, 35R11, 65L12, 76M22, 65L20

Key words: Time fractional fluid equation, finite difference, spectral method, stability, convergence.

1. Introduction

Non-Newtonian fluids find numerous applications in material processing, chemical and nuclear industries, geophysics, oil reservoir engineering and bio-engineering [17]. Blood, ketchup, shampoo, paint, polymer melts, drilling mud, various oils and greases are common examples of non-Newtonian fluids [16, 47]. A variety of fluids and industrial applications provide a major motivation for research in this field. An important class of non-Newtonian flows is represented by viscoelastic fluids, which includes second order fluids among others.

Investigating the viscosity of gases, Clerk Maxwell proposed the following fluid stress equation

$$(\text{stress}) + \tau \frac{d(\text{stress})}{dt} = (\text{viscosity}) \times \frac{d(\text{strain})}{dt}, \tag{1.1}$$

where τ is the relaxation time — cf. [31]. If straining motion is long compared to τ, the second term in the left-hand side of (1.1) can be dropped and the stress is proportional to the strain rate, which gives rise to the viscous behavior. On the other hand, for short

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time strain scale (relative to τ), the first term in the left-hand side of (1.1) can be removed, so that stress is proportional to the strain itself displaying elastic behavior. Note that phenomenological viscoelastic models are based on springs and dashpots. Springs obey Hooke's law whereas dashpots Trouton's (or Newton's) law and their combinations lead to various viscoelastic models such as Maxwell or Zener ones [7]. Modern constitutive equations for viscoelastic fluids are usually expressed in a covariant tensorial form following the principles of Oldroyd [5]. For the second order fluids, the Rivlin-Ericksen theory of differential type becomes popular among experimenters and theorists. In this model the stress response of a deforming fluid body is characterised by stretching and kinematic tensors — cf. Ref. [10].

Models based on ordinary differential equations have a relatively restricted class of solutions, which does not provide an adequate description of the complex systems in general. This problem can be overcome by using fractional equations. In particular, if a spring represents a zero order element and a dashpot a first order element, then a new component, called spring-pot, can be considered as the one of order $\beta \in [0,1]$. The parameter β can provide connection between pure elastic and viscous behaviors. Replacing the dashpot with a spring-pot, a modified fractional order viscoelastic model can be developed [28]. The current models of viscoelasticity involving fractional calculus are based on the formal replacement of the first-order derivatives in ordinary rheological constitutive equations by fractional derivatives of a non-integer order $\beta \in (0,1)$ [38]. However, this procedure does not always guarantee the resulting expression to be physically reasonable [14]. We note that rheological constitutive equations with fractional derivatives are actively used in the description of polymers and melts [12]. In some cases, the corresponding fractional equations are linked to molecular theories [2]. Besides, modified viscoelastic models properly describe the behavior of xanthan gum, sesbania gels and blood flows [44].

The study of fluid flows through porous media also attracted a lot of attention due to various industrial applications, including irrigation, food processing, soap and cellulose solutions, blood motion and crude oil recovery from reservoir rocks [20, 30, 32, 39]. On the other side, anomalous diffusion is present in surface growth, transport fluids in porous media [45], two-dimensional rotating flows [43], diffusion at liquid surfaces [6], plasma [8], subrecoil laser cooling [3], continuous time random walks [19, 29], non-Markovian dynamical processes in protein folding [33]. Anomalous diffusion deviates from the standard Fichean description of Brownian motion, where the squared mean displacement exhibits a nonlinear growth with respect to time — i.e. $< x^2 >\sim t^{\alpha}$. The parameter α belongs to the interval $(0,1)$ in the case of anomalous sub-diffusion and is greater than 1 for anomalous super-diffusion. The later may occur in chaotic or turbulent processes through enhanced transport of particles [18].

The goal of this paper is to study generalised second grade fluids with anomalous diffusion equation in porous media and to develop efficient numerical methods for its solution. The presence of fractional derivatives makes this task more complicated, since their point values depend on the global behaviour of the corresponding functions. Let us recall attempts for the generalised second grade fluid with Darcy's effect based on Wright function and Fox H-function [11, 48, 49]. Besides, Sohail *et al.* [42] considered spectral time