East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.300418.300618

Vol. **8**, No. 3, pp. 598-610
August 2018

The Numerical Study of the Ground States of Spin-1 Bose-Einstein Condensates with Spin-Orbit-Coupling

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Received 30 April 2018; Accepted (in revised version) 30 June 2018.

Abstract. A projection gradient method for computing the ground state of the spin-orbit-coupled spin-1 Bose-Einstein condensate at extremely low temperatures is proposed. The continuous gradient flows are discretised by a second-order finite difference method in space and the Crank-Nicolson method in time. Our discretisation preserves the total mass conservation and the energy diminishing property. Numerical results show the efficiency of the method.

AMS subject classifications: 65M10, 78A48

Key words: Spin-orbit-coupled spin-1 Bose-Einstein condensate, projection gradient method, ground state, energy functional minimisation.

1. Introduction

Bose-Einstein condensate (BEC) is a state of matter of the dilute boson gas cooled close to absolute zero temperature. In these conditions, a large fraction of bosons occupy the lowest quantum state [26]. It was first observed in experiments in 1995 and became an ideal test ground for the experimental study of condensed matter phenomena. In particular, since the spin-orbit coupling (SOC) is ubiquitous in nature, the realisation of spin-orbit interaction in cold atomic gases is a hot topic nowadays [1, 9, 38, 39, 49]. Thus the spin-orbit coupling has been successfully induced in recent experiments in a neutral atomic Bose-Einstein condensates by dressing two atomic spin states with a pair of lasers [21–23]. These experiments triggered a strong activity in the area of spin-orbit-coupled cold atoms

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and a number of exciting phenomena have been discovered [13,47]. The spin-1 BECs with isotropic spin-orbit coupling and rotation have been also studied — cf. Refs. [14,19,48]. It was found that SOC plays a crucial role in majorana fermions [45], spintronic devices [18], spin Hall effect [15] and topological insulators [10,27].

On the other hand, the creation of SOC in ultracold atomic gases attracted theoretical attention — cf. Refs. [6, 8, 11, 12, 21, 23, 29, 32, 35, 38, 47, 50]. In particular, BEC with various types of spin-orbit interaction has been considered in Refs. [20,24,34] and the spin-orbit-coupled BEC with distinct internal structures of bosons in Refs. [16, 17, 34, 36, 37]. Nevertheless, it is worth noting that although different couplings can generate non-trivial ground-state structures in spin-1/2, spin-1 and spin-2 BEC [4, 27, 33, 44, 46], there is no efficient numerical method to find such ground state solutions.

The projection gradient method (PGM), first used in nonlinear programming [30, 31], was later extended to functional minimisation problems with constraints [2, 25, 41, 42]. The key step in the method is the construction of a gradient flow projected into a feasible region or space. This approach has been recently combined with the conjugated gradient method [3]. Here we want to extend it onto energy functional minimisation with constraints and to use in the study of ground state solutions of the spin-orbit-coupled spin-1 BEC at extremely low temperatures. The method diminishes energy, conserves constraint during its implementation and evolves the continuous gradient flow to find the ground states.

This paper is organised as follows. In Section 2, we define the ground state solutions for spin-orbit coupled spin-1 BEC at very low temperatures and show that the ground state solutions satisfy the virial theorem. In Section 3, we use the projection gradient method to determine the ground state solutions of the spin-orbit-coupled spin-1 BEC and present two numerical methods for discretising the corresponding continuous gradient flows. In Section 4, we compare these numerical method and apply one of them to the ground state of the spin-orbit-coupled spin-1 BEC. Section 5 contains our conclusions and discussion.

2. Ground State of Spin-Orbit Coupled Spin-1 BEC

Let Ω be a bounded domain in $\mathbb{R}^d$. Using the physical Hamiltonian of the spin-orbit-coupled spin-1 BEC at very low temperature [11,33,43,46,47], we define the dimensionless energy functional of the spin-orbit-coupled spin-1 BEC by

$$
\begin{aligned}
&E(\phi_1,\phi_0,\phi_{-1})\\
&=\int_\Omega f(\phi_1,\bar{\phi}_1,\nabla\phi_1,\nabla\bar{\phi}_1,\cdots,\phi_{-1},\bar{\phi}_{-1},\nabla\phi_{-1},\nabla\bar{\phi}_{-1})d\mathbf{x}\\
&=\int_\Omega \Bigg\{\sum_{j=1,0,-1}\bar{\phi}_j h_d\phi_j+\frac{\beta_n}{2}\rho^2+\frac{\beta_s}{2}(\rho_1+\rho_0-\rho_{-1})\rho_1\\
&\quad+\frac{\beta_s}{2}(\rho_1+\rho_{-1})\rho_0+\frac{\beta_s}{2}(\rho_0+\rho_{-1}-\rho_1)\rho_{-1}+\beta_s\left(\bar{\phi}_{-1}\phi_0^2\bar{\phi}_1+\phi_{-1}\bar{\phi}_0^2\phi_1\right)
\end{aligned}
$$